Circling up the Wagons: Unifying Mathematics for the Calculus Student

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So often in the teaching of mathematics the material is presented in short, self-contained topics. Students in high school learn bits and pieces of mathematics in courses named Algebra I, Geometry, Algebra II, and Trigonometry. Then, at the college level and into graduate school, university curricula perpetuate the erroneous impression that mathematics is made up of a honeycomb of discrete topics. For instance, the course work at the master's degree level consists of courses titled Topology I and II, Real Analysis I and II, and Algebra I and II, further obscuring the fact that mathematics is a coherent whole which is formed by the beautiful interplay between these seemingly self-contained disciplines.

The academically naïve view of mathematics described above can be seen to be demonstrably false if one takes an objective inventory of the juncture in mathematics where the disciplines interact and merge. In fact, one need not look far in graduate mathematics for some simple examples where the line of demarcation between the disciplines begins to blur. For instance, the algebraic constructs of homological algebra are now so indispensable in the field of topology that simplicial and singular homology theory is taught in most master's level topology courses. In my own discipline of algebraic geometry, the interplay between complex analysis of one and several complex variables, partial differential equations, topology, and commutative algebra is ever present, and the lines between these disciplines along which the course work is traditionally divided are frequently breached.

Graduate students in mathematics are sufficiently well-versed in the subject to recognize and appreciate the interplay between the branches of mathematics. However, in the context of a liberal arts and sciences curriculum, which by its very nature emphasizes the unified nature of scholarship, is it possible to convey the unity, and therefore, the beauty of mathematics to undergraduate students whose training in the subject is limited? How do we, as teachers, communicate to our undergraduate students that mathematics is a single subject consisting of many contributing areas of study? Further, how do we demonstrate to our students that it is the creative nature of mathematics which is the conduit linking the individual disciplines?

For mathematicians like myself who are dedicated to scholarship in the tradition of a liberal arts and sciences education, the responsibility for presenting to undergraduates a unified view of our field lies squarely on our shoulders, and it is a responsibility which we should passionately embrace. The true challenge in designing such a presentation at the undergraduate level lies in the very fact that the students' exposure to mathematics is limited.

Early in my career, I taught a second semester calculus course at Brown University which included a standard discussion of parametric curves in the plane, unit tangent and normal vectors, and curvature; and an equally standard discussion of solutions to second order linear differential equations with constant coefficients and applications. At the end of the semester, in an attempt to unify the course, to demonstrate how mathematicians think, and to illustrate what types of questions mathematicians ask, I gave the following presentation. Keep in mind that this lecture is aimed at a mostly first-year class of undergraduates taking a second semester calculus course.

Let $\vec{r}(t) = \langle r\cos(t), r\sin(t)\rangle$ be a smooth parameterization of a circle of radius $r > 0$ in the plane. We compute the velocity by taking the derivative of this position vector:

$$\vec{v}(t) = \frac{d\vec{r}}{dt}(t) = \langle -r\sin(t), r\cos(t)\rangle$$

and we compute the speed by taking the norm of the velocity:

$$v(t) = \|\vec{v}(t)\| = \sqrt{(-r\sin(t))^2 + (r\cos(t))^2} = r.$$

The unit tangent vector $\vec{T}$ is the velocity divided by the speed:

$$\vec{T}(t) = \vec{v}/v = \langle \sin(t), \cos(t)\rangle.$$

Recall that if $s = s(t)$ is the arclength parameter, then ds/dt is the speed, v. By the Chain Rule, we then have

$$\begin{aligned} \frac{d\vec{T}}{ds} &= \frac{d\vec{T}}{dt} \div \frac{ds}{dt} \\ &= \langle -\cos(t), -\sin(t)\rangle \div v \\ &= \frac{1}{r}\langle -\cos(t), -\sin(t)\rangle. \end{aligned}$$

The curvature $\kappa(t)$ is then

$$\kappa(t) = \left\|\frac{d\vec{T}}{ds}\right\| = \frac{1}{r},$$

and the unit normal vector $\vec{N}(t)$ is obtained by dividing $d\vec{T}/ds$ by κ.

In particular, we note that the curvature of a circle is a constant function of t and that the value of this function is the reciprocal of the radius. It is also easily computed that a line has constant curvature equal to zero. Both these computations and the results are regularly presented in calculus courses.

At this point in the lecture I offered the class a question typical of one which a mathematician would ask: Is this a complete list of plane curves with constant curvature? That is, is every curve with constant curvature either a line or a circle?

By posing this question to the class, my intent was to illustrate the basic pattern of scholarly inquiry: It is this method of observing a result, posing a related question, and then attempting to answer the question, that forms the basic pattern of mathematical research, indeed, of all investigative scientific research.

Having studied second order linear ordinary differential equations earlier in the course, the class is now in a position to answer this question.

Proposition 1. *Every smooth plane curve with constant curvature is either a line or a circle.*

Proof. Let C a smooth plane curve with constant curvature κ and let $\vec{r}(s) = \langle x(s), y(s)\rangle$ be a parameterization of C with respect to arclength. Let $\vec{T}(s) = \langle \dot{x}(s), \dot{y}(s)\rangle$, where we use the standard physics notation for differentiation. By translating the coordinate axes, we may assume $\vec{r}(0) = \langle 0, 0\rangle$, and by rotating the coordinate axes about the origin, we may assume $\vec{T}(0) = \langle 0, 1\rangle$. Hence, $x(0) = y(0) = \dot{x}(0) = 0$ and $\dot{y}(0) = 1$.

By definition, $\kappa = \left\| d\vec{T}/ds \right\|$, so $\kappa = 0$ if and only if $d\vec{T}/ds \equiv 0$ as a function of s. This is equivalent to saying:

$$\begin{aligned} \ddot{x}(s) &= 0, \\ \ddot{y}(s) &= 0. \end{aligned}$$

Solving these differential equations with the initial conditions $x(0) = y(0) = \dot{x}(0) = 0$ and $\dot{y}(0) = 1$, yields

$$\begin{aligned} x(s) &= 0, \\ y(s) &= s, \end{aligned}$$

which is the line $x = 0$.

Now, assume κ is a positive constant. It follows that $d\vec{T}/ds$ is never zero, so the unit normal vector $\vec{N}(s)$ is always defined. By reflecting the coordinate axes around the y-axis, if necessary, we may assume $\vec{N}(0) = \langle -1, 0\rangle$ and still preserve our previous conditions that $\vec{r}(0) = \langle 0, 0\rangle$ and $\vec{T}(0) = \langle 0, 1\rangle$. We know that

$$\frac{d\vec{T}}{ds} = \kappa \vec{N}(s). \tag{1}$$

Differentiating equation (1) with respect to s and recalling that κ is constant, we obtain

$$\frac{d^2\vec{T}}{ds^2} = \kappa \frac{d\vec{N}}{ds}. \tag{2}$$

Since $\vec{N}(s)$ is a unit vector, $\vec{N} \cdot \vec{N} = 1$. By differentiating this equations with respect to s, we have $\frac{d\vec{N}}{ds} \cdot \vec{N} = 0$. So, $\frac{d\vec{N}}{ds}$ must be parallel to $\vec{T}(s)$. Hence, we may write

$$\frac{d\vec{N}}{ds} = \lambda \vec{T} \tag{3}$$

where $\lambda(s)$ is a real-valued function of s. We note that $\vec{N} \cdot \vec{T} = 0$, so

$$\begin{aligned} 0 &= \frac{d}{ds}(\vec{N} \cdot \vec{T}) \\ &= \frac{d\vec{N}}{ds} \cdot \vec{T} + \vec{N} \cdot \frac{d\vec{T}}{ds} \\ &= \frac{d\vec{N}}{ds} \cdot \vec{T} + \vec{N} \cdot (\kappa \vec{N}) \\ &= \frac{d\vec{N}}{ds} \cdot \vec{T} + \kappa. \end{aligned}$$

So, $\frac{d\vec{N}}{ds} \cdot \vec{T} = -\kappa$. Taking the dot product of equation (3) with $\vec{T}(s)$ yields

$$-\kappa = \frac{d\vec{N}}{ds} \cdot \vec{T} = \lambda \vec{T} \cdot \vec{T} = \lambda,$$

so $\lambda = -\kappa$. Substituting $\lambda = -\kappa$ into equation (3) and the result into equation (2) yields

$$\frac{d^2\vec{T}}{ds^2} + \kappa^2 \vec{T} = 0,$$

or, equivalently,

$$\begin{aligned} \frac{d^2\dot{x}}{ds^2} + \kappa^2 \dot{x} &= 0, \\ \frac{d^2\dot{y}}{ds^2} + \kappa^2 \dot{y} &= 0. \end{aligned}$$

Solving these second order linear homogeneous differential equations using standard techniques mastered by the students at this point yields

$$\begin{aligned} \dot{x}(s) &= c_1 \cos(\kappa s) + c_2 \sin(\kappa s), \\ \dot{y}(s) &= c_3 \cos(\kappa s) + c_4 \sin(\kappa s). \end{aligned}$$

Since

$$\vec{T}(0) = \langle \dot{x}(0), \dot{y}(0) \rangle = \langle 0, 1 \rangle,$$

we have $c_1 = 0$ and $c_3 = 1$. Since

$$\langle -1, 0 \rangle = \vec{N}(0) = \frac{1}{\kappa} \left. \frac{d\vec{T}}{ds} \right|_{s=0} = \frac{1}{\kappa} \langle \ddot{x}(0), \ddot{y}(0) \rangle,$$

we have $c_2 = -1$ and $c_4 = 0$. Hence

$$\begin{aligned} \dot{x}(s) &= -\sin(\kappa s), \\ \dot{y}(s) &= \cos(\kappa s). \end{aligned}$$

Integrating and using the fact that $\vec{r}(0) = \langle 0, 0 \rangle$, we get

$$\begin{aligned} x(s) &= \frac{1}{\kappa}\cos(\kappa s), \\ y(s) &= \frac{1}{\kappa}\sin(\kappa s). \end{aligned}$$

Hence, C is a circle of radius $1/\kappa$. □

The reader might consider this presentation beyond the scope of a typical second semester calculus course. If one means that a lecture of this type is not normally presented in a second semester calculus course, I must agree. Nevertheless, the course content had included all the elements needed in order to answer the questions posed, and from that perspective, the students had all the knowledge to understand the solution, although the construction of the solution required a level of mathematical maturity far beyond the students' abilities. This modest example demonstrates the interaction between the differential geometric notions of tangent vector, normal vectors, and curvature; and the techniques of ordinary differential equations. Further, the answer, albeit reached by elementary methods, has substantial mathematical significance in its own right: A plane curve is classified up to rigid motion of the plane by its curvature.

We as a nation are facing a growing crisis in mathematics. Fewer and fewer U.S. citizens are entering graduate studies in mathematics and the present demand for Ph.D.'s in mathematics already exceeds the supply. If this trend is to be reversed, undergraduates must be presented with mathematics as a field which is worthy of study in its own right. Further, the worth of mathematics is derived from the intellectual satisfaction of studying and later contributing to the academic depth of the subject. Mathematics will be able to attract quality students into graduate study only if we, as teachers, make the incredible richness of mathematics available to our undergraduates as early as possible in their careers. Perhaps the best way of doing this is to present mathematics as it truly is—a single discipline whose academic depth and intellectual complexity is provided by the beautiful and creative interaction of numerous specialized areas of study.