Ceva's Theorem and Its Applications

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Theorem 1 (Ceva's theorem)**.** *Let ΔABC be a triangle and let points D, E, F, be located on sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. The line segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent if and only if*

$$\frac{AE}{CE} \cdot \frac{CD}{BD} \cdot \frac{BF}{AF} = 1.$$

Proof. Construct ΔABC with points D, E, and F on sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively.

Suppose the line segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent. Let O be the point of intersection. Construct line ℓ through C parallel to line $\overline{AB}$ and extend segments $\overline{AD}$ and $\overline{BE}$ to meet ℓ at points G and H, respectively. See Figure **??**.

Since ℓ and $\overline{AB}$ are parallel, we have angle $\angle ABE$ congruent to $\angle CHE$, since they are alternate interior angles with respect to the transversal $\overline{BH}$. Similarly, $\angle BAE$ congruent to $\angle HCE$, since they are alternate interior angles with respect to the transversal $\overline{AC}$. Hence, triangles ΔABE and ΔCHE are similar.

By the same reasoning, we have the following pairs of similar triangles

$$\begin{aligned} \Delta ABE &\sim \Delta CHE \\ \Delta AOF &\sim \Delta GOC \\ \Delta BOF &\sim \Delta HOC \\ \Delta GDC &\sim \Delta ADB. \end{aligned}$$

From the first similarity, we have

$$\frac{AE}{CE} = \frac{AB}{CH}. \tag{1}$$

From the second and third similarities, we have

$$\frac{BF}{AF} = \frac{BF}{OF} \cdot \frac{OF}{AF} = \frac{CH}{OC} \cdot \frac{OC}{CG} = \frac{CH}{CG}. \tag{2}$$

From the fourth similarity, we have

$$\frac{CD}{BD} = \frac{CG}{AB}. \tag{3}$$

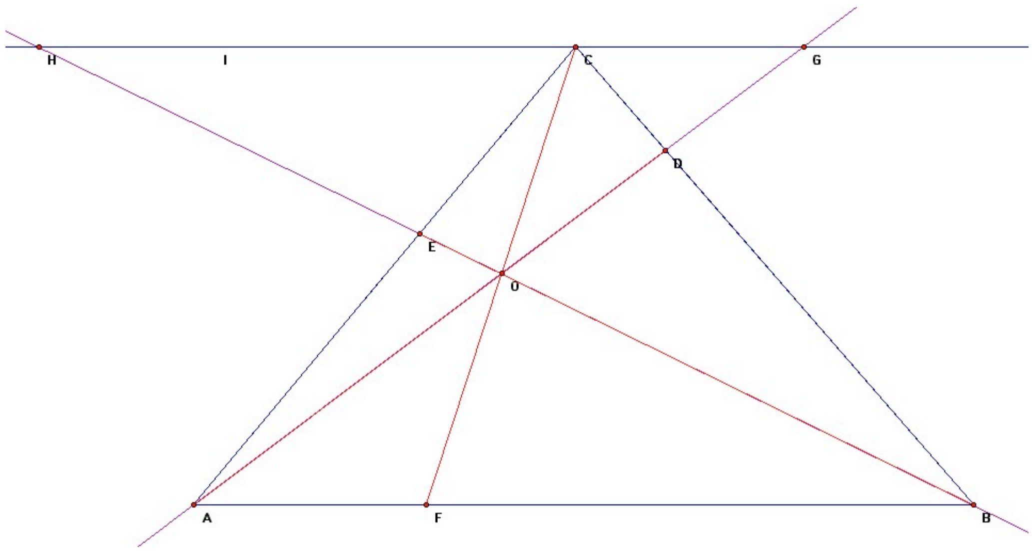

Figure 1.

Then, using the equations (**??**), (**??**), and (**??**), we have

$$\frac{AE}{CE} \cdot \frac{CD}{BD} \cdot \frac{BF}{AF} = \frac{AB}{CH} \cdot \frac{CG}{AB} \cdot \frac{CH}{CG} = 1.$$

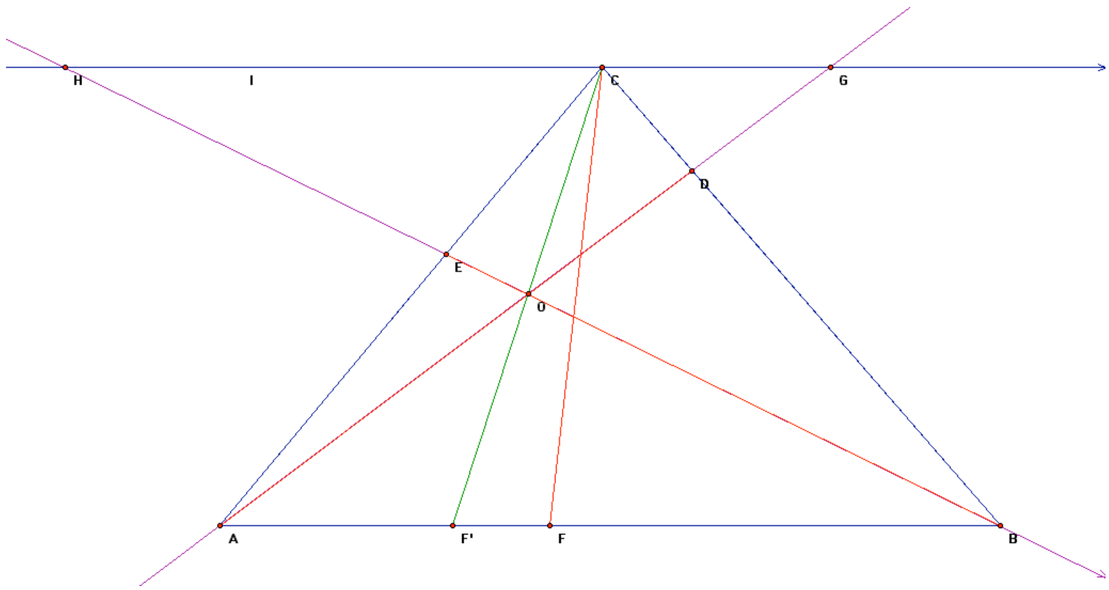

Figure 2.

Conversely, suppose

$$\frac{AE}{CE} \cdot \frac{CD}{BD} \cdot \frac{BF}{AF} = 1. \tag{4}$$

Let O be the intersection of $\overline{AD}$ and $\overline{BE}$. Construct the segment $\overline{OC}$ and extend this segment to meet $\overline{AB}$ at a point F'. See Figure **??**.

Since the segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF'}$ are concurrent, by the first part of the proof, we have

$$\frac{AE}{CE} \cdot \frac{CD}{BD} \cdot \frac{BF'}{AF'} = 1. \tag{5}$$

From equations (**??**) and (**??**), it follows that

$$\begin{aligned}
\frac{BF}{AF} &= \frac{BF'}{AF'} \\
\frac{BF}{AF} + 1 &= \frac{BF'}{AF'} + 1 \\
\frac{BF}{AF} + \frac{AF}{AF} &= \frac{BF'}{AF'} + \frac{AF'}{AF'} \\
\frac{AF + FB}{AF} &= \frac{AF' + F'B}{AF'} \\
\frac{AB}{AF} &= \frac{AB}{AF'} \\
AF &= AF'.
\end{aligned}$$

Hence, $F = F'$ and the three segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent. $\square$

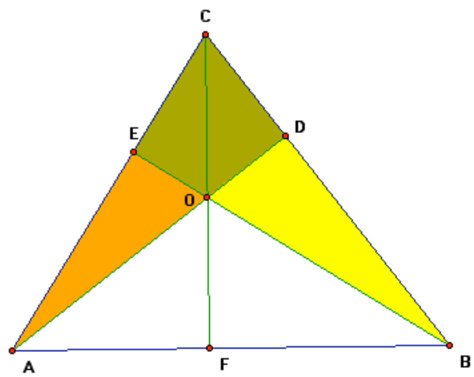

Figure 3.

Corollary 1. *The three medians of a triangle are concurrent.*

Proof. For the medians of a triangle, we have $AE = EC$, $BD = DC$, $AF = BF$, so this result follows immediately from Ceva's theorem. $\square$

Definition 1. *The point of intersection of the three medians of a triangle is called the **centroid** of the triangle.*

Corollary 2. *The three altitudes of a triangle are concurrent.*

Proof. Construct ΔABC with points D, E, and F the feet of the altitudes from vertices A, B, and C, respectively.

Consider the right triangles ΔBEC and ΔADC. See Figure **??**. Since $\angle BCE$ is an acute angle in the right triangle ΔBEC, angles $\angle BCE = \angle ACD$ and $\angle CBE$ are complementary. In the right triangle ΔADC, angles $\angle ACD = \angle BCE$ and $\angle CAD$ are complementary. So, angles $\angle CBE$ and $\angle CAD$ are congruent. Hence, ΔBEC is similar to ΔADC, whereby

$$\frac{EC}{DC} = \frac{BC}{AC}. \tag{6}$$

Consider the right triangles ΔFAC and ΔEAB. See Figure **??**. Since $\angle FCA$ is an acute angle in the right triangle ΔFAC, angles $\angle FCA$ and $\angle CAF = \angle EAB$ are complementary. In right triangle ΔEAB, angles $\angle EBA$ and $\angle EAB = \angle CAF$ are complementary. So, angles $\angle FCA$ and $\angle EBA$ are congruent. Hence, ΔEAB is similar to ΔFAC. Hence,

$$\frac{AE}{AF} = \frac{AB}{AC}. \tag{7}$$

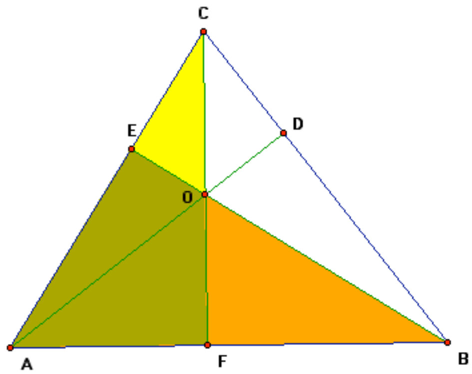

Figure 4.

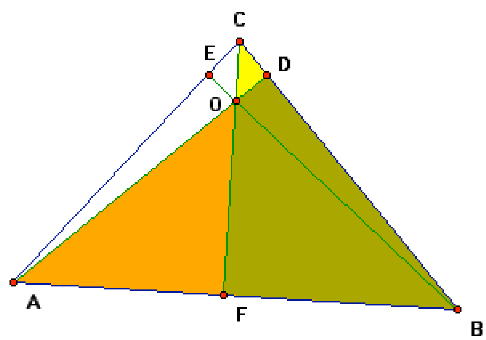

Figure 5.

Consider the right triangles ΔBFC and ΔBDA. See Figure **??**. Since $\angle BCF$ is an acute angle in the right triangle ΔBFC, angles $\angle CBF = \angle DBA$ and $\angle BCF$ are complementary. In right triangle ΔBDA, angles $\angle DAB$ and $\angle DBA = \angle CBF$ are complementary. So, angles $\angle BCF$ and $\angle DAB$ are congruent. Hence, ΔBFC is similar to ΔBDA. Hence,

$$\frac{BF}{BD} = \frac{BC}{AB}. \tag{8}$$

By equations (**??**), (**??**), and (**??**),

$$\begin{aligned} \frac{AE}{CE} \cdot \frac{CD}{BD} \cdot \frac{BF}{AF} &= \frac{AE}{AF} \cdot \frac{CD}{EC} \cdot \frac{BF}{BD} \\ &= \frac{AB}{AC} \cdot \frac{AC}{BC} \cdot \frac{BC}{AB} = 1, \end{aligned}$$

so, by Ceva's theorem, the three altitudes are concurrent. $\square$

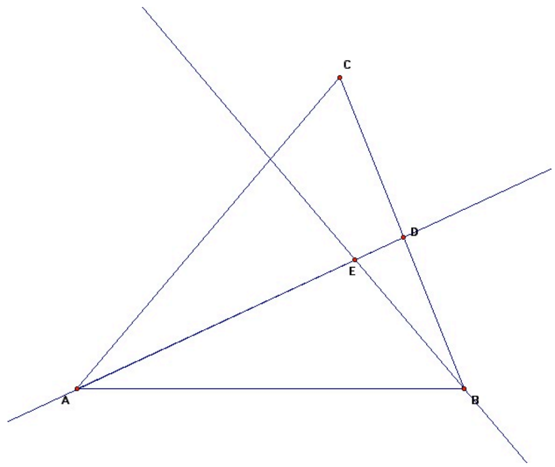

Figure 6.

Definition 2. *The point of intersection of the three altitudes of a triangle is called the* ***orthocenter*** *of the triangle.*

Lemma 1. *Let ΔABC be a triangle and let the angle bisector of angle $\angle CAB$ intersect side $\overline{BC}$ in the point D. Then*

$$\frac{AB}{AC} = \frac{BD}{CD}.$$

That is, the ratio of the segments into which an angle bisector divides the opposite side equals the ratio of the two respective adjacent sides of the triangle.

Proof. Let ΔABC be a triangle and let the angle bisector of $\angle CAB$ meet side $\overline{BC}$ in the point D. Construct an angle with angle measure equal to that of angle $\angle ACB$ with vertex at B, one side on $\overline{AB}$, and the other side on the C side of line $\overline{AB}$. Let E be the point of intersection of the terminal side of this constructed angle with the segment $\overline{AD}$, extended, if necessary. See Figure **??**. The angles $\angle ACB = \angle ACD$ and $\angle ABE$ are congruent by construction and angles $\angle CAD$ and $\angle DAB = \angle BAE$ are congruent by hypothesis. Hence, triangle ΔABE and ΔACD are similar. So,

$$\frac{AB}{BE} = \frac{AC}{CD} \tag{9}$$

Since the triangles ΔABE and ΔACD, are similar, their corresponding pairs of angles are congruent. Hence, $\angle AEB \cong \angle ADC$. Taking supplements, we have that $\angle BDE \cong \angle BED$. That is, ΔBED is isosceles, whereby $BE = BD$. Substituting this into equation (**??**), we get

$$\frac{AB}{BD} = \frac{AC}{CD}.$$

as desired. ∎

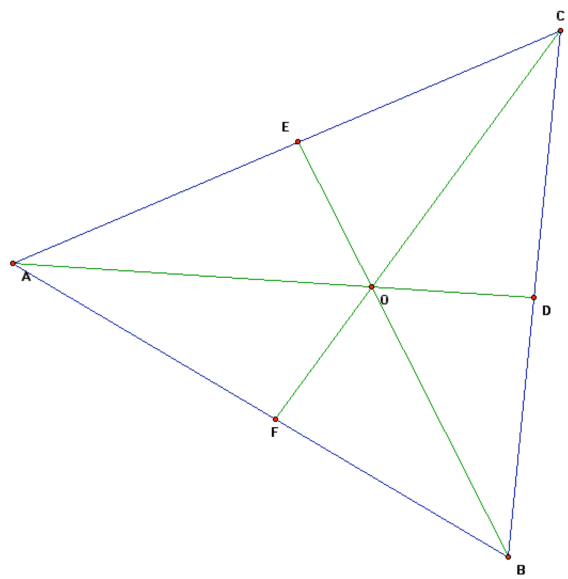

Figure 7.

Corollary 3. *The three angle bisectors of a triangle are concurrent.*

Proof. Construct ΔABC with points D, E, and F be the points of intersection of the angle bisectors with the sides opposite A, B, and C, respectively. See Figure **??**.

Applying the Lemma **??** three times, we have

$$\begin{aligned} \frac{AE}{EC} &= \frac{AB}{BC} \\ \frac{CD}{BD} &= \frac{AC}{AB} \\ \frac{BF}{AF} &= \frac{CB}{AC} \end{aligned}$$

Hence,

$$\frac{AE}{CE} \cdot \frac{CD}{BD} \cdot \frac{BF}{AF} = \frac{AB}{BC} \cdot \frac{AC}{AB} \cdot \frac{CB}{AC} = 1,$$

so the three angle bisectors are concurrent, by Ceva's Theorem. ☐

Definition 3. *The point of intersection of the three angle bisectors of a triangle is called the* ***incenter*** *of the triangle. This is the center of the inscribed circle of the triangle.*

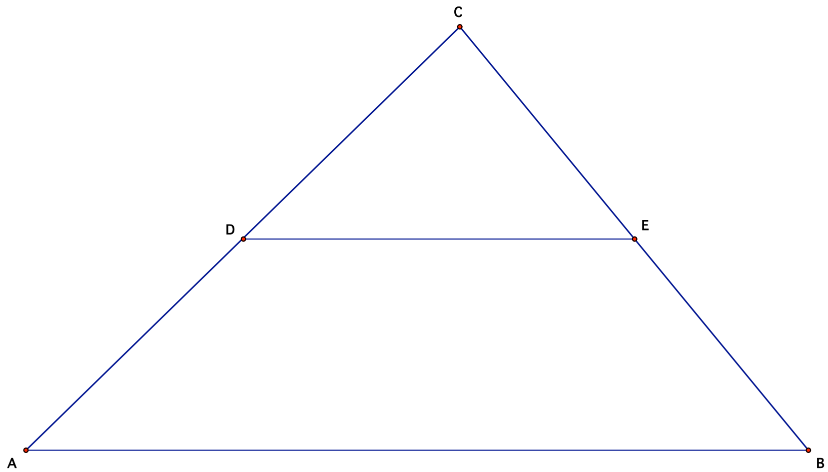

Figure 8.

Lemma 2. *Let ΔABC be a triangle and let D and E be the midpoints of sides $\overline{AC}$ and $\overline{BC}$. Then segment $\overline{DE}$ is parallel to segment $\overline{AB}$. (See Figure* **??***.)*

Proof. Let ΔABC be a triangle and let points D and E be the midpoints of segments $\overline{AC}$ and $\overline{BC}$. Extend $\overline{DE}$ past E to a point F, so that $\overline{DE} \cong \overline{EF}$. See Figure **??**. Since D and E are the midpoints of segments $\overline{AC}$ and $\overline{BC}$, respectively, we have $\overline{AD} \cong \overline{CD}$ and $\overline{BE} \cong \overline{EC}$. Since the vertical angles $\angle CED$ and $\angle BEF$ are congruent, we have that triangles ΔCED and ΔBEF are congruent by SAS. Hence, $\overline{CD}$ and $\overline{BF}$ are congruent. By the transitivity of congruence, $\overline{AD}$ and $\overline{BF}$ are congruent.

Secondly, since ΔCED and ΔBEF are congruent, $\angle CDE \cong \angle BFE$. Since these two angles are alternate interior angles with respect to the transversal $\overline{DF}$, it follows that segments $\overline{AD}$ and $\overline{BF}$ are parallel.

Construct segment $\overline{BD}$. Since $\overline{BD}$ is a transversal to the parallel segments $\overline{AD}$ and $\overline{BF}$, $\angle FBD$ and $\angle ADB$ are congruent.

Since $\overline{AD}$ and $\overline{BF}$ are congruent, $\overline{BD}$ is congruent to itself, and $\angle FBD$ and $\angle ADB$ are congruent, triangles ΔABD and ΔFDB are congruent by SAS. Hence, $\angle ABD$ and $\angle FDB$ are congruent. Since these are alternate interior angles with respect to the transversal $\overline{BD}$ and the segments $\overline{DE}$ and $\overline{AB}$, we see that these two segments must be parallel. $\square$

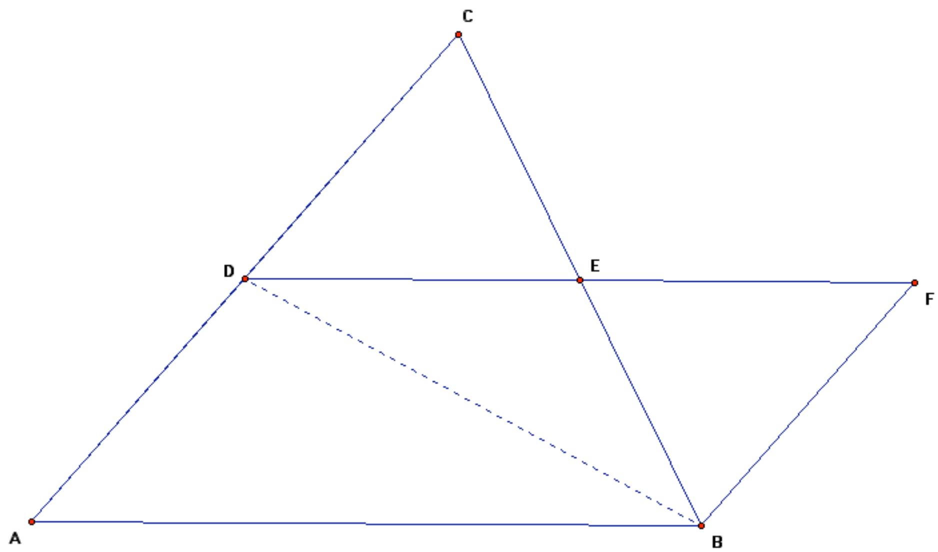

Figure 9.

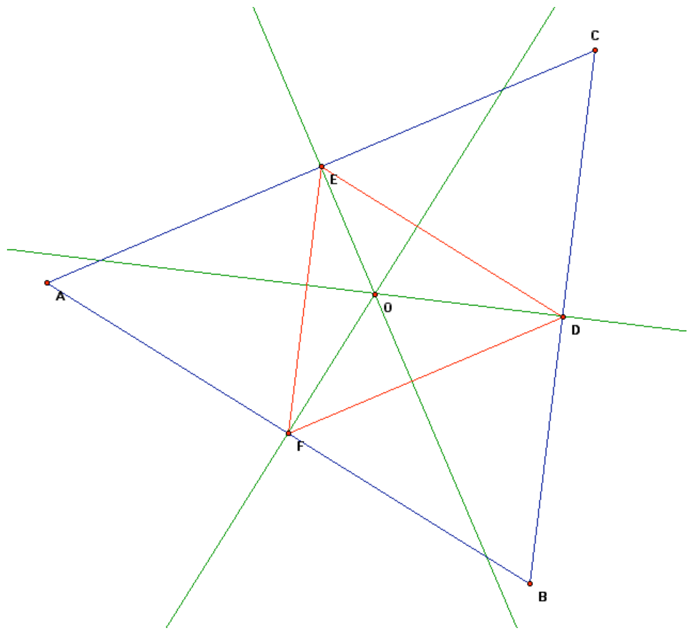

Figure 10.

Corollary 4. *The three perpendicular bisectors of a triangle are concurrent.*

Proof. Let ΔABC be a triangle. Construct the midpoints D, E, F of the three sides of ΔABC, $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Construct line segments $\overline{DE}$, $\overline{DF}$, and $\overline{EF}$. See Figure **??**.

By Lemma **??**, segment $\overline{DE}$ is parallel to segment $\overline{AB}$, segment $\overline{EF}$ is parallel to segment $\overline{BC}$, and segment $\overline{DF}$ is parallel to segment $\overline{AC}$. It follows that the perpendicular bisectors of the sides of ΔABC are perpendicular to the respective sides of ΔDEF. Hence, the perpendicular bisectors of the sides of the triangle ΔABC are the altitudes of the triangle ΔDEF. Since we already know that the altitudes of a triangle are concurrent, we conclude that the perpendicular bisectors of ΔABC are concurrent. $\square$

Definition 4. *The point of intersection of the three perpendicular bisectors of a triangle is called the* ***circumcenter*** *of the triangle. It is the center of the circumscribed circle of the triangle.*

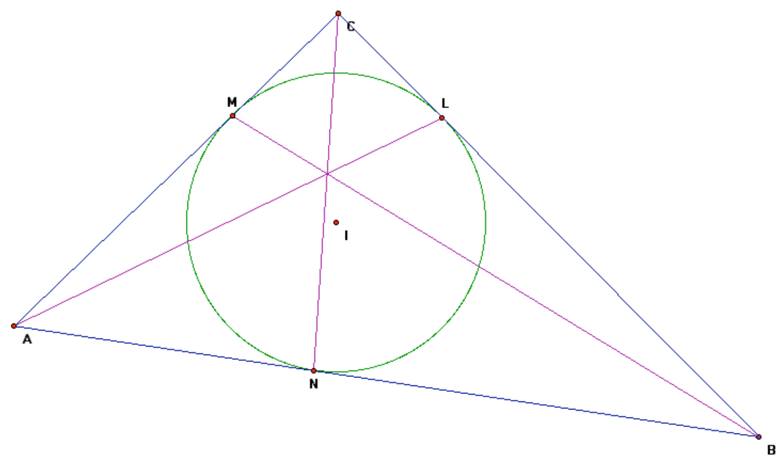

Figure 11.

Corollary 5. *Let ΔABC be a triangle and let L, M, and N, be the points of tangency of the inscribed circle to the sides of the triangle opposite vertices A, B, and C, respectively. The line segments $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$ are concurrent.*

Proof. Let ΔABC be a triangle. Construct the inscribed circle to ΔABC and let L, M, and N, be the points of tangency of the inscribed circle to the sides of the triangle opposite vertices A, B, and C, respectively. See Figure **??**. Since $\overline{AM}$ and $\overline{AN}$ are common external tangent segments to the inscribed circle, they are congruent. Similarly, $\overline{BL}$ and $\overline{BN}$ are congruent and $\overline{CL}$ and $\overline{CM}$ are congruent. Hence,

$$\frac{AM}{MC} \cdot \frac{CL}{BL} \cdot \frac{BN}{AN} = \frac{AM}{AN} \cdot \frac{CL}{CM} \cdot \frac{BN}{BL} = 1 \cdot 1 \cdot 1 = 1.$$

By Ceva's Theorem, the segments $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$ are concurrent. $\square$

Definition 5. *The point of intersection of the three segments from the vertices of a triangle to the points of tangency on the opposite sides of the triangle is called the* ***Gergonne point*** *of the triangle.*