Basic Algebraic Geometry, Volume 2
Chapter 5, Section 1: The Spectrum of a Ring

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1. Definition of Spec(A)

(1) The set of prime ideals of A is its **prime spectrum** or simply **spectrum**, and denoted by $\mathrm{Spec}(A)$. Prime ideals are called points of $\mathrm{Spec}(A)$.

(2) The ring itself is NOT a prime ideal.

(3) Every nonzero ring has at least one maximal ideal.

(4) A ring homomorphism $\varphi : A \to B$. The inverse of any prime ideal B is a prime ideal of A. This defines a map ${}^a\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ called the **associated map**.

(5) Work through the map $\mathrm{Spec}(\mathbb{C}[T]) \to \mathrm{Spec}(\mathbb{R}[T])$ associated with the inclusion $\mathbb{R}[T] \hookrightarrow \mathbb{C}[T]$.

(6) Work through the map $\mathrm{Spec}(\mathbb{Z}[i]) \to \mathrm{Spec}(\mathbb{Z})$ associated with the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$.

(7) Work through the map $\mathrm{Spec}(\mathbb{Z}[T]) \to \mathrm{Spec}(\mathbb{Z})$ associated with the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[T]$.

(8) There is a map $\varphi : A \to A_S$ defined by $a \mapsto a/1$, and hence a map ${}^a\varphi : \mathrm{Spec}(A_S) \to \mathrm{Spec}(A)$. This map is inclusion and the image U_S is the set of prime ideals disjoint from S. The inverse map $\psi : U_S \to \mathrm{Spec}(A_S)$ is of the form

$$\psi(\mathfrak{p}) = \mathfrak{p}A_S = \{x/s \mid x \in \mathfrak{p} \text{ and } s \in S\}.$$

If $f \in A$ and $S = \{f^n \mid n = 0, 1, \dots\}$ then A_S is denoted A_f.

2. Properties of Points of $\mathrm{Spec}(A)$

(1) Each $x \in \mathrm{Spec}(A)$ has a field of fractions $k(x)$, the **residue field** of x. There is a map $A \to k(x)$ whose kernel is the prime ideal corresponding to the point $x \in \mathrm{Spec}(A)$. We write $f(x)$ for the image of $f \in A$ in $k(x)$ under this map. This is the value of f at x. So, each element of A gives a function on $\mathrm{Spec}(A)$, but the values of the function lie in different fields depending on x.

(2) The element $f \in A$ is not uniquely determined by its corresponding function on $\mathrm{Spec}(A)$. For example, any f in the nilradical of A is identically zero on $\mathrm{Spec}(A)$, but is not 0 in A.

Proposition. *An element $f \in A$ is contained in every prime ideal of A if and only if it is nilpotent (that is, $f^n = 0$ for some n).*

(3) For each point $x \in \mathrm{Spec}(A)$ there is a local ring $\mathcal{O}_x$. A point $x \in \mathrm{Spec}(A)$ is **regular** (or **simple**) if the local ring $\mathcal{O}_x$ is Noetherian and is a regular local ring. A Noetherian local ring A of dimension d with maximal ideal $\mathfrak{m}$ is regular if $\dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right) = d$. This means $\mathfrak{m}$ can be generated by d elements.

(4) Suppose $A = k[X]$ and $x \in \mathrm{Spec}(A)$ be a nonclosed point. What is the geometric meaning of a prime ideal $x \in \mathrm{Spec}(A)$ being regular? This means the subvariety $Y \subset \mathrm{Spec}(A)$ is not contained in the subvariety of singular points of X.

(5) Let $\mathfrak{m}_x$ be the maximal ideal of the local ring $\mathcal{O}_x$ of a point $X \in \mathrm{Spec}(A)$. Then $\mathcal{O}_x/\mathfrak{m}_x = k(x)$, and the group $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over $k(x)$. If $\mathcal{O}_x$ is Noetherian, then this space a finite dimensional. The dual vector space

$$\Theta_x = \mathrm{Hom}_{k(x)}(\mathfrak{m}/\mathfrak{m}^2, k(x))$$

is the **tangent space** to $\mathrm{Spec}(A)$ at x.

(6) For $A = \mathbb{Z}$, each closed point has 1-dimenional tangent space, so each closed point is regular.

(7) Consider $A = \mathbb{Z}[mi] = \mathbb{Z}[y]/(y^2 + m^2) = \mathbb{Z} + \mathbb{Z}mi$, where $m > 1$ is an integer and $i^2 = -1$. The inclusion $\varphi : A \hookrightarrow A' = \mathbb{Z}[i]$ defines a map

$$^a\varphi : \mathrm{Spec}(A') \to \mathrm{Spec}(A). \tag{1}$$

If we look at prime ideals coprime to m then this is a one-to-one correspondence and the corresponding local rings are equal. Hence a point $x \in \mathrm{Spec}(A')$ is not regular only if the prime ideal divides m. Primes ideals of A' dividing m are in one-to-one correspondence with prime factors p of m, and are given by $\mathfrak{p} = (p, mi)$. In this case,

$k(x) = \mathbb{F}_p$ is the field of p elements, and $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{p}/\mathfrak{p}^2$ is a 2-dimensional $\mathbb{F}_p$-vector space, since $\mathfrak{p}^2 \subset (p)$. Hence $\mathfrak{m}_x$ is not principal, and the local ring $\mathcal{O}_x$ is not regular. Thus all the prime ideals $\mathfrak{p} = (p, mi)$ with $p \mid m$ are singular points of Spec(A). The map (1) is a resolution of these singularites.

3. The Zariski Topology of Spec(A)

(1) Any set $E \subset A$ defined a subset $V(E) \subset \mathrm{Spec}(A)$ consisting of prime ideals $\mathfrak{p}$ containing E.

(2) We have

$$V\left(\bigcup_\alpha E_\alpha\right) = \bigcap_\alpha V(E_\alpha)$$
$$V(I) = V\left(E'\right) \cup V\left(E''\right), \text{ where } I = (E') \cap (E'')$$

This shows that the set $V(E)$ satisfy the axioms for the closed sets of a topological space.

(3) The topology on Spec(A) in which the $V(E)$ are the closed sets is called the **Zariski topology** or **spectral topology**.

(4) For a homomorphism $\varphi : A \to B$ and any set $E \subset A$, we have $({}^a\varphi)^{-1}(V(E)) = V(\varphi(E))$. So, ${}^a\varphi$ is a continuous map.

(5) The natural homomorphism $\varphi : A \to A/\mathfrak{a}$ for an ideal $\mathfrak{a} \subset A$. Then ${}^a\varphi$ is a homeomorphism of Spec($A/\mathfrak{a}$) to the closed set $V(\mathfrak{a})$. So, every closed subset of Spec(A) is isomorphic to Spec of a ring.

(6) For $S \subset A$ a multiplicatively closed set, let

$$\varphi : A \to A_S, \quad U_S = {}^a\varphi(\mathrm{Spec}(A_S)) \quad \text{and} \quad \psi : U_S \to \mathrm{Spec}(A_S)$$

be as before. Give $U_S \subset \mathrm{Spec}(A)$ the subspace topology. Then ψ is also continuous, so Spec(A_S) is homeomorphic to the subspace $U_S \subset \mathrm{Spec}(A)$.

(7) Special case: $S = \{f^n \mid n = 0, 1, \dots\}$ with $f \in A$ and element that is not nilpotent. Then $U_S = \mathrm{Spec}(A) \setminus V(f)$, where $V(f) = V(E)$ with $E = \{f\}$. The open sets of the form $\mathrm{Spec}(A) \setminus V(f)$ are called *prinicipal open sets*. They are denoted $D(f)$. They form a basis for the Zariski topology. The prinicipal open sets $D(f)$ are homeomorphic to Spec(A_f).

(8) $\mathrm{Spec}(A)$ is compact.

(9) The Zariski topology is very nonclassical on $\mathrm{Spec}(A)$. Not only is it not Hausdorff, it actually has non-closed points.

(10) $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. So the closed points in $\mathrm{Spec}(A)$ are the maximal ideals in A. If A is a domain, then $(0) \in \mathrm{Spec}(A)$ is an everywhere dense point.

(11) x is a **specialization** of y if $x \in \overline{\{y\}}$. An everywhere dense point if called a **generic point** of the space. $\mathrm{Spec}(A)$ has a generic point if the nilradical is prime.

(12) Let $\mathcal{O}_x$ be the local ring of a nonsingular point of an algebraic curve. Then (0) is the generic point of $\mathrm{Spec}(\mathcal{O}_x)$ and is open. The maximal ideal $\mathfrak{m}_x$ is a closed point.

4. Irreducibility, Dimension

(1) A topological space X is **reducible** if $X = X_1 \cup X_2$ where $X_1, X_2 \subsetneq X$ are closed sets.

(2) $\mathrm{Spec}(A)$ is irreducible if and only if it has a generic point if and only if $\mathfrak{N}(A)$ is a prime ideal.

(3) Since every closed subset of $\mathrm{Spec}(A)$ is Spec of quotient of A, there is a one-to-one correspondence between points and irreducible subsets of $\mathrm{Spec}(A)$ given by sending a point to its closure.

(4) If A is a Noetherian ring, then there exists a decomposition

$$\mathrm{Spec}(A) = X_1 \cup \cdots \cup X_r,$$

where X_i are irreudicble closed subsets and $X_i \not\subset X_j$ for $i \neq j$ and this decomposition is unique.

(5) If $A = A_1 \oplus \cdots \oplus A_r$, then $\mathrm{Spec}(A) = \coprod_i \mathrm{Spec}(A_i)$.

(6) Take $A = \mathbb{Z}[\sigma] = \mathbb{Z} + \mathbb{Z}\sigma$, with $\sigma^2 = 1$. The nilradical of A is (0), but it's not a prime ideal.

$$\mathrm{Spec}(A) = X_1 \cup X_2, \text{ where } X_1 = V(1+\sigma) \text{ and } X_2 = V(1-\sigma). \tag{1}$$

The homomorphisms $\varphi_1, \varphi_2 : A \to \mathbb{Z}$ with kernels $(1+\sigma)$ and $(1-\sigma)$ define homeomorphisms

$$\begin{aligned} {}^a\varphi_1 &: \mathrm{Spec}(\mathbb{Z}) \to V(1+\sigma) \\ {}^a\varphi_2 &: \mathrm{Spec}(\mathbb{Z}) \to V(1-\sigma) \end{aligned}$$

which show that X_1 and X_2 are irreducible. So that (1) is a decomposition of Spec(A) into irreducible components.

The intersection $X_1 \cap X_2$ is a point x_0. All the points unequal to x_0 are regular while x_0 is singular. x_0 is a "double point with distinct tangents" of Spec(A).

We can picture Spec(A) using the map ${}^a\varphi : \mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{Z})$ where $\mathbb{Z} \hookrightarrow A$ is the natural inclusion. See Figure 1.

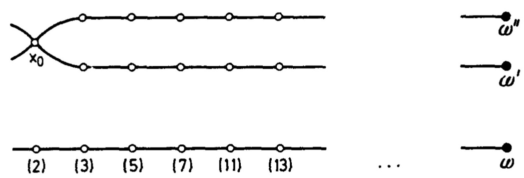

Figure 1: Picture of Spec(A)

(7) The **dimension** of a topological space X is the number n such that X has a chain of irreducible closed sets

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n,$$

and no such chain with more than n terms.

Propositon A. If A is a Noetherian local ring, then the dimension of Spec(A) is finite, and equal to the Krull dimension of A.

Propositon B. A ring that is finitely generated over a ring having finite dimension is again finite dimensional.

Propositon C. If A is Noetherian then

$$\dim (A[T_1, \ldots, T_n]) = \dim (A) + n.$$