Basic Algebraic Geometry, Volume 1
Chapter 1, Section 5: Products and Maps of Quasiprojective Varieties

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1. Products

(1) If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are quasiprojective subvarieties of affine spaces, then

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

is a quasiprojective variety in $\mathbb{A}^n \times \mathbb{A}^m$. This is the **product** of X and Y. If X and Y are replaced by isomorphic varieties X' and Y', then $X \times Y \cong X' \times Y'$.

(2) Consider $\mathbb{P}^N$ with coordinates w_{ij} having two subscripts, $0 \le i \le n$, $0 \le j \le m$. So, $N = (n+1)(m+1) - 1$. If $x = (u_0 : \cdots : u_n) \in \mathbb{P}^n$ and $y = (v_0 : \cdots : v_m) \in \mathbb{P}^m$, we set

$$\varphi(x, y) = (w_{ij}), \quad \text{with } w_{ij} = u_i v_j \text{ for } 0 \le i \le n, 0 \le j \le m.$$

The defining equations are

$$w_{ij} w_{k\ell} = w_{kj} w_{i\ell} \text{ for } 0 \le i, k \le n \text{ and } 0 \le j, \ell \le m.$$

(3) If $w_{00} \ne 0$, we have

$$x = (w_{00} : \cdots : w_{n0}) \quad \text{and} \quad y = (w_{00} : \cdots w_{0m}).$$

So, φ is an embedding of $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^N$ with image the subvariety $W \subset \mathbb{P}^N$.

(4) Consider open sets $\mathbb{A}_0^n \subset \mathbb{P}^n$ given by $u_0 \ne 0$ and $\mathbb{A}_0^m \subset \mathbb{P}^m$ given by $v_0 \ne 0$, and $\mathbb{A}_{00}^N \subset \mathbb{P}^N$ by $w_{00} \ne 0$. Let $x_i = u_i/u_0$, $y_i = v_i/v_0$, and $z_i = w_{ij}/w_{00}$ be affine coordinates on $\mathbb{A}_0^n, \mathbb{A}_0^m$, and $\mathbb{A}_{00}^N$. Then $\varphi(\mathbb{A}_0^n \times \mathbb{A}_0^m) = W \cap \mathbb{A}_{00}^N = W_{00}$. On W_{00} we have $z_{i0} = x_i$, $z_{0j} = y_j$ and $z_{ij} = z_{i0} z_{0j}$ for $i, j > 0$. It follows from this that $\varphi(\mathbb{P}^n \times \mathbb{P}^m) \cap \mathbb{A}_{00}^N = W_{00}$ is isomorphic to $\mathbb{A}^{n+m}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_m)$, and φ defines an isomorphism $\mathbb{A}_0^n \times \mathbb{A}_0^m \to W_{00}$.

The embedding $\varphi : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ is the **Segre embedding**, and the image $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^N$ the **Segre variety**.

(5) The Segre variety is a determinantal variety. The vanishing of 2×2 minors in the matrix (w_{ij}).

(6) When $n = m = 1$, the Segre variety has one equation: $w_{11}w_{00} - w_{01}w_{10} = 0$. So, $\varphi(\mathbb{P}^n \times \mathbb{P}^m)$ is a nondegenerate quadric surface $Q \subset \mathbb{P}^3$.

For $\alpha = (\alpha_0, \alpha_1) \in \mathbb{P}^1$, the set $\varphi(\alpha \times \mathbb{P}^1)$ is the line in $\mathbb{P}^3$ given by $\alpha_1 w_{00} = \alpha_0 w_{10}$, As α runs through $\mathbb{P}^1$, these lines give all the generators of one of the two families of lines on Q. Likewise, $\varphi(\mathbb{P}^1 \times \beta)$ is a line of $\mathbb{P}^3$, and as β runs through $\mathbb{P}^1$, these lines give the generators of the other family.

(7) The product $X \times Y$ of two quasiprojective varieties in projective space by way of the Segre embedding.

(8)

Theorem. *A subset $X \subset \mathbb{P}^n \times \mathbb{P}^m$ is a closed algebraic subvariety if and only if it is given by a system of equations*

$$G_k(u_0 : \cdots : u_n; v_0 : \cdots : v_m) = 0 \quad \textit{for } k = 1, \ldots, t,$$

homogeneous separately in each set of variables u_i and v_j. Every closed algebraic subvariety of $\mathbb{P}^n \times \mathbb{A}^m$ is given by a system of equations

$$g_k(u_0 : \cdots : u_n; y_1 : \cdots : y_m) = 0 \quad \textit{for } k = 1, \ldots, t,$$

that are homogeneous in $u_0, \ldots, u_n$.

(9) This extends to products with more than two factors. Then the polynomials must be homogeneous in each group of variables.

2. The Image of a Projective Variety is Closed

Mark, put proofs in this section.

(1)

Theorem. *The image of a projective variety under a regular map is closed.*

(2)

Lemma. *The graph of a regular map is closed in $X \times Y$.*

The graph is the inverse image of Δ under the regular map $(f, i) : X \times Y \to Y \times Y$.

(3)

Theorem. *If X is a projective variety, and Y is a quasiprojective variety, the second projection $p : X \times Y \to Y$ takes closed sets to closed sets.*

Put proof here.

(4) Theorem (1) extends to maps $f : X \to Y$ between quasiprojective varieties, namely those that factor as a composite of a **closed embedding** $\iota : X \hookrightarrow \mathbb{P}^n \times Y$ (that is, an isomorphism of X with a closed subvariety) and the projection $p : \mathbb{P}^n \times Y \to Y$. Such maps are called **proper**.

(5)

Corollary. *If φ is a regular function on an irreducible projective variety then $\varphi \in k$.*

View φ as a map $f : X \to \mathbb{A}^1$. The image has to be closed and since X is irreducible, this forces the image to be a point.

(6)

Corollary. *A regular map $f : X \to Y$ from an irreducible projective variety X to an affine variety Y maps X to a point.*

Each component function is a regular map from an irreducible projective variety X to an affine variety $Y \subset \mathbb{A}^n$, so each component function is constant.

(7) Consider the representation of forms of degree m in $n+1$ variables as points in $\mathbb{P}^N$ with $N = \binom{m+n}{m} - 1$.

Proposition. *Points $\xi \in \mathbb{P}^N$ corresponding to reducible homogeneous polynomials F form a closed set.*

3. Finite Maps

Mark, put proofs in this section.

(1) Let $f : X \to Y$ be a regular map of affine varieties such that $f(X)$ is dense in Y. Then f^* defines an inclusion $k[Y] \hookrightarrow k[X]$.

Definition. f is a finite map if $k[X]$ is integral over $k[Y]$.

(2) If f is a finite map, each point has only finitely many inverse images.

(3)

Theorem. *A finite map is surjective.*

(4)

Lemma. *If a ring B is a finite A-module where $A \subset B$ is a subring containing 1_B, then for an ideal $\mathfrak{a}$ of A,*

$$\mathfrak{a} \subsetneqq A \Rightarrow \mathfrak{a}B \subsetneqq B$$

(5)

Corollary. *A finite map takes closed sets to closed sets.*

(6) Finiteness is a local property.

Theorem. *If $f : X \to Y$ is a regular map of affine varieties, and every point $x \in Y$ has an affine neighborhood $U \ni x$ such that $V = f^{-1}(U)$ is affine and $f : V \to U$ is finite, then f itself is finite.*

(7)

Definition. A regular map $f : X \to Y$ of quasiprojective varieties is **finite** if any point $y \in Y$ has an affine neighborhood V such that the set $U = f^{-1}(V)$ is affine and $f : U \to V$ is a finite map between affine varieties.

(8) This property has important consequences that relate to arbitrary maps.

Theorem. *If $f : X \to Y$ is a regular map and $f(X)$ is dense in Y then $f(X)$ contains an open set of Y.*

(9)

Theorem. *If $X \subset \mathbb{P}^n$ is a closed subvariety disjoint from a d-dimensional linear subspace $E \subset P^n$ then the projection $\pi : X \to \mathbb{P}^{n-d-1}$ with center E defines a finite map $X \to \pi(X)$.*

(10)

Theorem. *Suppose $F_0, \dots, F_s$ are forms of degree m on $\mathbb{P}^n$ having no common zeros on a closed variety $X \subset \mathbb{P}^n$. Then*

$$\varphi(x) = (F_0(x) : \cdots : F_s(x))$$

defines a finite map $\varphi : X \to \varphi(X)$.

4. Noether Normalization

(1)

Theorem. *For an irreducible projective variety X there exists a finite map $\varphi : X \to \mathbb{P}^n$ to a projective space.*

(2)

Theorem. *For an irreducible affine variety X there exists a finite map $\varphi : X \to \mathbb{A}^m$ to an affine space.*