# Basic Algebraic Geometry, Volume 1
# Chapter 1, Section 4: Quasiprojective Varieties

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## 1. Closed Subsets of Projective Space

(1) Let $V$ be a vector space of dimension $n+1$ over a field $k$. The set of one-dimensional subspaces of $V$ is *$n$-dimensional projective space*, denoted $\mathbb{P}(V)$ or $\mathbb{P}^n$.

(2) Definition of homogeneous coordinates

(3) If $f \in k[S_0, \dots, S_n]$ vanishes at $\xi \in \mathbb{P}^n$ for all homogenous coordinates for $\xi$, then each homogeneous components vanish at $\xi$.

(4) A set $X \subset \mathbb{P}^n$ is a *closed subset* if it consists of all points at which a finite number of polynomials with coefficients in $k$ vanish. A closed subset defined by one homogeneous equation $F = 0$ is called a *hypersurface*. The degree of the polynomial is the *degree* of the hypersurface. A hypersurface of degree 2 is called a *quadric*.

(5) The set of polynomials $f \in k[S_0, \dots, S_n]$ that vanish at all points $x \in X$ is the *ideal* of $X$, denoted $\mathfrak{A}_X \subset k[S_0, \dots, S_n]$. This ideal is *homogeneous* or *graded*. In particular, any closed projective set can be defined by a finite set of homogeneous equations.

(6) Any homogeneous ideal $\mathfrak{A} \subset k[S_0, \dots, S_n]$ defines a closed subset $X \subset \mathbb{P}^n$.

(7) The Grassmannian. $\mathrm{Grass}(r, V)$ is the algebraic set consisting of all $r$-dimensional subspaces in $V$. Consider $\bigwedge^r V$ and send a basis $\{f_1, \dots, f_r\}$ for an $r$-plane to the element $f_1 \wedge \cdots \wedge f_r$. Choosing another basis changes the image by a nonzero multiplicative constant, so the image of each $r$-plane in $V$ determines a point in $\mathbb{P}(\bigwedge^r V)$. The coordinates of the image of an $r$-plane are the *Plücker coordinates* of the $r$-plane.

(8) Let $u \in V^*$. For $x \in \bigwedge^1 V = V$ the *convolution* $u \lrcorner x$ is just $u(x)$. For $x \in \bigwedge^0 V = k$ we set $u \lrcorner x = 0$. For any $x \in \bigwedge^r V$ the convolution $u \lrcorner x = 0$ can be extended in a unique way from $x \in \bigwedge^1 V = V$ if we require the property

$$u \lrcorner (x \wedge y) = (u \lrcorner x) \wedge y + (-1)^a (x \wedge (u \lrcorner y)) \quad \text{for } x \in \bigwedge^a V.$$

Here, $u \lrcorner \bigwedge^r V \subset \bigwedge^{r-1} V$. The element $u \lrcorner x$ for $u \in V^*$ and $x \in \bigwedge^r V$ is called the *convolution* of $u$ and $x$. Finally, for $u_1, \dots, u_s \in V^*$ the element $u_1 \lrcorner (u_2 \lrcorner \cdots \lrcorner (u_s \lrcorner x) \cdots)$ depends only on $x$ and $y = u_1 \wedge \cdots \wedge u_s \in \bigwedge^s V^*$, and is denoted by $y \lrcorner x$. Here $y \lrcorner x \in \bigwedge^{r-s} V$ if $r \geq s$ and $y \lrcorner x = 0$ if $r < s$.

(9) Generally, not every point in $\mathbb{P}(\bigwedge^r V)$ is the image of an $r$-plane. The conditions for $x \in \bigwedge^r V$ to be of the form $x = f_1 \wedge \cdots \wedge f_r$ are given by

$$(y \lrcorner x) \wedge x = 0 \quad \text{for all } y \in \bigwedge^{r-1} V^*.$$

Let $\{u_i\}$ the dual basis for $V^*$ to the basis $\{e_i\}$ of $V$, then these equations can be written in coordinates. They take the form

$$\sum_{t=1}^{r+1} (-1)^t p_{i_1 \dots i_{r-1} j_t} p_{j_1 \dots \widehat{j_t} \dots j_{r+1}} = 0$$

for all sequences $i_1, \dots i_{r-1}$ and $j_1, \dots j_{r+1}$.

The variety defined in $\mathbb{P}(\bigwedge^r V)$ by these equations is called the *Grassmannian*, denoted by $\mathrm{Grass}(r, V)$ or $\mathrm{Grass}(r, n)$ where $n = \dim(V)$.

We need to reconstruct $L$ from $P(L)$. Suppose $p_{1 \dots r} \neq 0$. If $p = (p_{i_1 \dots i_r}) = P(L)$, then $L$ has a basis of the form

$$f_i = e_i + \sum_{k > r} a_{ik} e_k \quad \text{for } i = 1, \dots, r.$$

where $a_{ik} = (-1)^k p_{1 \dots \widehat{k} \dots rk}$, where we have set $p_{1 \dots r} = 1$ for convenience.

(10) The first nontrivial case is when $r = 2$. Then

$$(u \lrcorner x) \wedge x = \frac{1}{2}(u \lrcorner (x \wedge x)) \quad \text{for } u \in V^* \text{ and } x \in \bigwedge^2 V.$$

This reduces to $u \lrcorner (x \wedge x) = 0$ for all $u \in V^*$. That is, simply

$$x \wedge x = 0.$$

(11) When $n = 4$ we have $\dim \bigwedge^4 V = 1$, so this reduces to a single equation in the Plücker coordinates $p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}$:

$$p_{12} p_{34} - p_{13} p_{24} + p_{14} p_{23} = 0. \tag{1}$$

Planes $L \subset V$ in a 4-dimensional space $V$ correspond to lines $\ell \subset \mathbb{P}(V)$ in projective 3-space. In this case, coordinates in $V$ are denoted by $x_0, x_1, x_2, x_3$ and the Plücker coordinates $p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$, and (1) takes the form

$$p_{01} p_{23} - p_{02} p_{13} + p_{03} p_{12} = 0. \tag{2}$$

This is a quadric in projective 5-space $\mathbb{P}\left(\bigwedge^2 V\right)$.

(12) Determinantal varieties. Quadratic forms in $n$ variables form a vector space $V$ of dimension $\binom{n+1}{2} = \frac{1}{2}n(n+1)$. Quadrics in an $(n-1)$ dimensional projective space are parametrized by point of the product space $\mathbb{P}(V)$. Among these, the degenerate quadrics are characterized by $\det(f) = 0$, where $f$ is the corresponding quadratic form. This is a hypersurface $X_1 \subset \mathbb{P}(V)$.

The quadrics of rank $\le n-k$ correspond to points of a set $X_k$ defined by setting all $(n-k+1) \times (n-k+1)$ minors of the matrix of $f$ to 0. A set of this type is called a *determinantal variety.* Another type of determinantal variety $M_k$ is defined in the space $\mathbb{P}(V)$, where $V$ is the space of $n \times m$ matrices, by the condition that a matrix has rank $\le k$.

(13) A homogeneous ideal $\mathfrak{A} \subset k[S]$ defines the empty set if and only if it contains the ideal $I_s$ for some $s > 0$.

(14) Closed subsets of projective space are projective closed sets, just as closed subsets of affine space are affine closed sets. For $Y \subset X$ both closed sets, $X \setminus Y$ is a open set in $X$.

(15) The set $\mathbb{A}^n_i = \{(\xi_0, \dots, \xi_n\} \mid \xi_i \neq 0\} \subset \mathbb{P}^n$ is in one-to-one corresponding with $\mathbb{A}^n$ and $\mathbb{P}^n = \bigcup_{i=0}^n \mathbb{A}^n_i$. The set $\mathbb{A}^n_i$ is an affine piece of $\mathbb{P}^n$.

(16) For any projective closed set $X \subset \mathbb{P}^n$, and any $i = 0, \dots, n$, the set $U_i = X \cap \mathbb{A}^n_i$ is open in $X$. It is a closed subset of $\mathbb{A}^n_i$.

(17) If $X$ is given by a system of homogeneous equations $F_0 = \cdots = F_m = 0$ and $\deg F_j = n_j$, then, for example, $U_0$ is given by the system

$$S_0^{-n_j} F_j = F_j(1, T_1, \dots, T_n) = 0, \quad \text{for } j = 1, \dots, m,$$

where $T_i = S_i/S_0$ for $i = 0, \dots, n$. The $U_i$ are the *affine pieces* of $X$. Obviously $X = \cup_i U_i$.

(18) A closed subset $U \subset \mathbb{A}^n_0$ defines a closed projective set $\overline{U}$ called its *projective completion*; $\overline{U}$ is the intersection of projective closed sets containing $U$. If $F(T_1, \dots, T_n)$ is any polynomial in the ideal $\mathfrak{A}$ of $U$ of degree $\deg F = k$, then the equations of $\overline{U}$ are of the form $S_0^k F(S_1/S_0, \dots, S_n/S_0)$. It follows that

$$U = \overline{U} \cap \mathbb{A}^n_0. \tag{3}$$

(19) For the time being, a *quasiprojective variety* is an open subset of a closed projective set. Both projective closed sets and affine closed sets are quasiprojective varieties.

(20) A *closed subset* os a quasiprojective variety $X \subset \mathbb{P}^n$ is its intersection with a closed set of projective space. Open set and neighborhood are defined similarly. The notion of irreducible variety and decomposition a variety into irreduicible components carries over word-for-word from the case of affine sets.

(21) From now on we use *subvariety* $Y$ of a quasiprojective variety $X \subset \mathbb{P}^n$ to mean any subset $Y \subset X$ which is itself a quasiprojective variety in $\mathbb{P}^n$. This is equivalent to saying that $Y = Z \smallsetminus Z_1$ with $Z$ and $Z_1 \subset X$ closed subsets.

## 2. Regular Functions

(1) If $X \subset \mathbb{P}^n$ is a quasiprojective variety, $x \in X$ and $f = P/Q$ is a homogenous function of degree 0 with $Q(x) \neq 0$, then $f$ defines a function on a neighborhood of $x$ in $X$ with values in $k$. We say $f$ is *regular* in a neighborhood of $x$, or simply at $x$. A function that is regular at all points $x \in X$ is a *regular function* on $X$. All regular functions on $X$ form a ring, that we denote by $k[X]$.

For $X$ a closed subset of affine space, this is the same definition as before.

(2) If $X$ is an irreducible projective set, $k[X]$ consists only of constants. [Proof later]

(3) OTOH, if $X$ is quasiprojective, $k[X]$ may turn out to be unexpectedly large. There are quasiprojective varieties for which $k[X]$ is not finitely generated.

(4) A map $f : X \to Y$ between quasiprojective varieties, with $Y \subset \mathbb{P}^m$, is *regular* if for every point $x \in X$ and for some affine piece $\mathbb{A}^m_i$ containing $f(x)$ there exists a neighborhood $U$ containing $x$ such that $f(U) \subset \mathbb{A}^m_i$ and the map $f : U \to \mathbb{A}^m_i$ is regular.

This is independent of the choice of affine piece $\mathbb{A}^m_i$.

(5) A regular map $f : X \to Y$ defines a homomorphism $f^* : k[Y] \to k[X]$.

(6) In practice $f(x) = (F_0(x) : \cdots : F_m(x)) \in \mathbb{P}^m$ where $F_0, \ldots, F_m$ are forms of the same degree and $F_i(x) \neq 0$ for some $i$.

(7) Two different formulas

$$f(x) = (F_0(x) : \cdots : F_m(x)) \text{ and } g(x) = (G_0(x) : \cdots : G_m(x)) \tag{1}$$

define the same map if and only if

$$F_i G_j = F_j G_i \text{ on } X \quad \text{for } 0 \le i, j \le m. \tag{2}$$

(8) A *regular map* $f : X \to \mathbb{P}^m$ of an irreducible quasiprojective variety $X$ to projective space $\mathbb{P}^m$ is given by an $(m+1)$-tuple of forms

$$(F_0 : \cdots : F_m) \tag{3}$$

of the same degree in the homogeneous coordinates of $x \in \mathbb{P}^n$, so that $F_i(x) \neq 0$ for at least one $i$. Then we write $f(x) = (F_0(x) : \cdots : F_m(x))$. Two maps (1) are considered equal if (2) holds.

(9) An *isomorphism* of varieties is a regular map having an inverse regular map.

(10) A quasiprojective variety $X'$ isomorphic to a closed subset of an affine space will be called an *affine variety*. The quasiprojective variety $X = \mathbb{A}^1 \setminus 0$ is an affine variety even though it is not a closed set in $\mathbb{A}^1$. The set $X$ is isomorphic to the hyperbola $xy = 1$ which is a closed set in $\mathbb{A}^2$.

(11) A quasiprojective variety isomorphic to a closed projective set will be called a *projective variety*. It is a fact that if $X \subset \mathbb{P}^n$ is a projective variety then it is closed in $\mathbb{P}^n$. So the notions of closed projective set and projective variety coincide and are both invariant under isomorphism.

(12) There are quasiprojective varieties that are neither affine nor projective.

(13) The property that a subset $Y \subset X$ is closed in a quasiprojective variety $X$ is a local property.

(14) Every point $x \in X$ has a neighborhood isomorphic to an affine variety.

(15) An open set $D(f) = X \setminus V(f)$ consisting fo the points of an affine variety $X$ such that $f(x) \neq 0$ is called a **principal open set**. Also $k[D(f)] = k[X][f^{-1}]$.

(16) Under any regular map, the inverse image $f^{-1}(Z)$ under any regular map $f : X \to Y$ os any closed subset $Z \subset Y$ is closed in $X$.

(17) The inverse image of any open set under a regular map is open. That is, regular maps are continuous.

(18) Any function $\varphi$ regular in a neighborhood of $f(x) \in Y$, the function $f^*(\varphi)$ is regular in a neighborhood of $x$.

## 3. Rational Functions

(1) Let $X \subset \mathbb{P}^n$ be an irreducible quasiprojective variety. Write $\mathcal{O}_X$ for the set of rational functions $f = P/Q$ in homogeneous coordinates $S_0, \ldots, S_n$ such that $P$, $Q$ are forms of the same degree and $Q \notin \mathfrak{A}_X$. $\mathcal{O}_X$ is the ring of **rational functions** on $X$.

(2) The functions $f \in \mathcal{O}_X$ with $P \in \mathfrak{A}_X$ is the maximal ideal $M_X$ in $\mathcal{O}_X$. The quotient ring $\mathcal{O}_X/M_X$ is the function field of $X$, and denoted $k(X)$.

(3) If $U$ is an open subset of an irreducible quasiprojective variety $X$, then $k(X) = k(U)$. In particular, $k(X) = k(\overline{X})$. So, in discussing function fields, we can restrict to affine or projective varieties.

(4) If $X$ is an affine variety, this definition agrees with the earlier one.

(5) We say $f \in k(X)$ is **regular** at a point $x \in X$ if it can be written in the form $f = F/G$, with $F$ and $G$ homogeneous of the same degree and $G(X) \neq 0$. The value $f(x) = F(x)/G(x)$ is the **value** of the function at $x$. The set of points at which a rational function $f$ is regular is a nonempty, open subset of $X$, called the **domain of definition**.

(6) A rational function can also be defined as a function regular on some open set $U \subset X$.

(7) A rational map $f : X \to \mathbb{P}^m$ is defined by giving $m+1$ forms $(F_0 : \cdots : F_m)$ of the same degree in the $n+1$ homogeneous coordinates of $\mathbb{P}^n$ containing $X$. At least one of these forms must not vanish on $X$.

(8) Two maps $(F_0 : \cdots : F_m)$ and $(G_0 : \cdots : G_m)$ are equal if $F_i G_j = F_j G_i$ on $X$ for all $i, j$.

(9) If we divide by one of the coordinate functions, we can define a rational map by $m+1$ rational functions on $X$, then the same notion of equality of maps.

(10) A rational map $f$ defined by functions $(f_0 : \cdots : f_m)$ such that all $f_i$ are regular at $x \in X$ and not all zero at $x$, then $f$ is regular at $x$. If defines a regular map on some neighborhood of $x \in \mathbb{P}^m$.

(11) The set where a rational function is regular is open. We can define a rational map to be a function regular on some open set $U \subset X$ on which $f$ is regular.

(12) Let $Y \subset \mathbb{P}^m$ is a quasiprojective variety and $f : X \to \mathbb{P}^m$, we say $f : X \to Y$ if there is an open set $U \subset X$ on which $f$ is regular and $f(U) \subset Y$. The union $\widetilde{U}$ of all such open sets is the **domain of definition** of $f$, and $f(\widetilde{U}) \subset Y$ is the **image** of $X$ in $Y$.

(13) If the image of a rational map $f : X \to Y$ is dense, rational map defines and inclusion of fields $f^* : k(Y) \to k(X)$.

(14) If a rational map $f : X \to Y$ has an inverse rational map then $f$ is **birational** or is a **birational equivalence**, and $X$ and $Y$ are birational. In this case, $f^* : k(Y) \to k(X)$ is an isomorphism.

(15) **Proposition**: Two irreducible varieties $X$ and $Y$ are birational if and only if they contain isomorphic open subsets $U \subset X$ and $V \subset Y$.

# 4. Examples of Regular Maps

(1) Projection with center $E \subset \mathbb{P}^n$, $\dim(E) = d$. $E$ given by $n-d$ linear equations $L_1 = L_2 = \cdots = L_{n-d}$. Then

$$\pi : \mathbb{P}^n \to \mathbb{P}^{n-d-1}$$
$$\pi(x) = (L_1(x) : \cdots : L_{n-d}(x))$$

Take any $(n-d-1)$-dimensional linear subspace $H \subset \mathbb{P}^n$ disjoint from $E$. Then there is a unique $(d+1)$-dimensional linear subspace $(E, x)$ passing through $E$ and any point $x \in \mathbb{P}^n \smallsetminus E$. This subspace intersects $H$ in a unique point, which is $\pi(x)$.

If $X$ intersects $E$ but is not contained in it, then projection from $E$ is a rational map on $X$.

(2) The Veronese embedding $\nu_{n,m} : \mathbb{P}^n \to \mathbb{P}^N$ where $N = \binom{n+m}{m} - 1$.

The regular map $v_m : \mathbb{P}^n \to \mathbb{P}^N$ defined by $v_{i_0 \cdots i_n} = u_0^{i_0} \cdots u_n^{i_n}$ for $\sum i_j = m/$. This is the $m^{\text{th}}$ **Veronese embedding** of $\mathbb{P}^n$ and the image is the **Veronese variety**. This is a determinantal variety cut out by quadrics.

If $F$ is a form of degree $m$ in the homogeneous coordinates of $\mathbb{P}^n$ and $H \subset \mathbb{P}^n$ is the hypersuface defined by $F = 0$, then $v_m(H) \subset v_m(\mathbb{P}^n) \subset \mathbb{P}^N$ is the intersection of $v_m(\mathbb{P}^n)$ with a hyperplane of $\mathbb{P}^N$. The Vernoese embedding allows us to reduce the study of some problems ocncerning hypersurfaces of degree $m$ to the case of hyperplanes.

The $m^{\text{th}}$ Vernoese image of the porjective line $v_m(\mathbb{P}^1) \subset \mathbb{P}^m$ is called the Veronese curve, the twisted $m$-ic curve, of the **rational normal curve** of degree $m$.