Basic Algebraic Geometry, Volume 1
Chapter 1, Section 3: Rational Functions

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1. Irreducible Algebraic Sets

(1) A closed algebraic set X is *reducible* if there exist proper closed subsets $X_1, X_2 \subsetneq X$ such that $X = X_1 \cup X_2$. Otherwise X is irreducible.

(2) Any closed set X is a finite union of irreducible closed sets. (Proof: Use Noetherian property.)

(3) If $X = \cup_i X_i$ is an expression of X as a finite union of irreducible closed sets, that if $X_i \subsetneq X_j$ for all $i \neq j$, we say such a representation is *irredundant*, and the X_i are the *irreducible components* of X.

(4) The irredundant representation of X as a finite union of irreducible closed sets is unique.

(5) A closed set X is irreducible if and only if its coordinate ring $k[X]$ has no zerodivisors. That is, if and only if $\mathfrak{A}_X$ is a prime ideal.

(6) If $Y \subset X$ are closed subsets, then the components of Y are contained in the components of X. The irreducibility of $Y \subset X$ is reflected in $\mathfrak{A}_Y \subset k[X]$ being a prime ideal.

(7) A hypersurface $X \subset \mathbb{A}^n$ with equation $f = 0$ is irreducible if and only if f is irreducible.

(8) A product of irreducible closed sets is irreducible.

2. Rational Functions

(1) If a closed set X is irreducible then the field of fractions of the coordinate ring $k[X]$ is the *function field* or *field of rational functions* of X, denoted $k(X)$.

(2) The field $k(X)$ consists of rational functions $F(T)/G(T)$ such that $G(T) \notin \mathfrak{A}_X$ and $F/G = F_1/G_1$ if $FG_1 - F_1G \in \mathfrak{A}_X$. This means $k(X)$ can be contructed as follows. Consider the subring $\mathscr{O}_X \subset k(T_1, \dots, T_n)$ of rational functions $f = P/Q$ with $P, Q \in k[T]$ and $Q \notin \mathfrak{A}_X$.

The functions f with $P \in \mathfrak{A}_X$ form an ideal M_X and $k(X) = \mathscr{O}_X/M_X$.

This confuses polys in $k[T_1, \dots, T_n]$ and regular functions in $k[X]$.

(3) A rational function may not define a function on all of X.

(4) A rational function $\varphi \in k(X)$ is *regular* at $x \in X$ if it can be written in the form $\varphi = f/g$ with $f, g \in k[X]$ and $g(x) \neq 0$. Then the element $f(x)/g(x) \in k$ is the *value* of φ at x.

(5) A rational function φ that is regular at all points of a closed subset X is a regular function on X.

(6) If φ is a rational function on a closed set X, the set of points at which φ is regular is nonempty and open. The set U where φ is regular is called the *domain of definition* of φ.

(7) For any finite system $\varphi_1, \dots, \varphi_m$ of rational functions, the set of points $x \in S$ at which they are all regular is again open and nonempty. Open is clear. Nonempty since X is irreducible.

(8) A rational function $\varphi \in k(X)$ is uniquely determined if it is specified on some nonempty open subset $U \subset X$. (Suppose $\varphi(x) = 0$ for all $x \in U$ and $\varphi \neq 0$ on X. If $\varphi = f/g$ with $f, g \in k[X]$. Then $X = X_1 \cup X_2$ where $X_1 = X \setminus U$ and X_2 is given by $f = 0$. This contradicts the fact that X is irreducible.)

3. Rational Maps

(1) For $X \subset \mathbb{A}^n$ an irreducible closed set, a *rational map* $\varphi : X \dashrightarrow \mathbb{A}^m$ is a map given by an m-tuple of rational functions $\varphi_1, \dots, \varphi_m \in k(X)$. The function φ is defined on some open subset of X.

(2) A *rational map* $\varphi : X \dashrightarrow Y$ to a closed subset $Y \subset \mathbb{A}^m$ is an m-tuple of rational functions $\varphi_1, \dots, \varphi_m \in k(X)$ such that, for all points $x \in X$ at which all the φ_i are regular, $\varphi(x) \in Y$. The *image of* X is the set

$$\varphi(X) = \{\varphi(x) \mid x \in X \text{ and } \varphi \text{ is regular at } x\}.$$

(3) The rational map φ is defined on an open set $U \subset X$.

(4) Rational functions $\varphi_1, \dots, \varphi_m \in k(X)$ define a rational map $\varphi : X \dashrightarrow Y$, we need $\varphi_1, \dots, \varphi_m$, as elements of $k(X)$, to satisfy all the equations on Y. Indeed, if this property holds then for any polynomial $u(T_1, \dots, T_m) \in \mathfrak{A}_Y$ the function $u(\varphi_1, \dots, \varphi_m) = 0$ on X. Then at each point x where all the φ_i are regular, we have $u(\varphi_1(x), \dots, \varphi_m(x)) = 0$ for all $u \in \mathfrak{A}_Y$. That is $(\varphi_1(x), \dots, \varphi_m(x)) \in Y$.

Conversely, if $\varphi : X \dashrightarrow Y$ is a rational map, then for every $u \in \mathfrak{A}_Y$ the function $u(\varphi_1, \dots, \varphi_m) \in k(X)$ vanishes on some nonempty open set $U \subset X$, and so is 0 on the whole of X. It follows that $u(\varphi_1, \dots, \varphi_m) = 0 \in k(X)$.

Does the above make sense?

(5) If $\varphi : X \dashrightarrow Y$ is a rational map and $\varphi(X)$ is dense in Y, then $\varphi^* : k(Y) \to k(X)$ is an isomorphic inclusion. ($\varphi(X)$ being dense in Y makes the homomorphism $k[Y] \to k(X)$ injective. Thus, we are able to extend this homomorphism to $k(Y)$.)

(6) Given two rational maps $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$ such that $\varphi(X)$ is dense in Y then we can define a composition map $\psi \circ \varphi : X \dashrightarrow Z$. If, in addition, $\psi(Y)$ is dense in Z, then so is $(\psi \circ \varphi)(X)$. Then the inclusions satisfy $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

(7) A rational map $\varphi : X \dashrightarrow Y$ is *birational* or is a *birational equivalence* if φ has an inverse rational map $\psi : Y \dashrightarrow X$, that is $\varphi(X)$ is dense in Y and $\psi(Y)$ is dense in X, and $\psi \circ \varphi = 1$ and $\varphi \circ \psi = 1$, where defined. In that case we say that X and Y are *birational* or *birationally equivalent*.

(8) The closed sets X and Y are birational iff the fields $k(X)$ and $k(Y)$ are isomorphic over k.

(9) A closed set that is birational to an affine space $\mathbb{A}^n$ is *rational*.

(10)

Examples 1. (a) Isomorphic sets are birational.

(b) An irreducible quadric $X \subset \mathbb{A}^n$ is rational.

(c) The hypersurface $X \subset \mathbb{A}^3$, $\mathrm{char}(k) \neq 3$, given by $x^3 + y^3 + z^3 = 1$ contains several lines, for example the two skew lines L_1 and L_2 defined by

$$L_1 : x + y = 0, z = 1, \qquad \text{and} \qquad L_2 : x + \varepsilon y = 0, z = \varepsilon,$$

where $\varepsilon \neq 1$ is a cube root of 1. This hypersurface is also rational.

Choose some plane $E \subset \mathbb{A}^3$ not containing L_1 or L_2. For $x \in X \setminus (L_1 \cup L_2)$, it is easy to verify that there is a unique line L passing through x and intersecting L_1 and L_2. Write $f(x)$ for the point of intersection $L \cap E$; then $x \mapsto f(x)$ is the required rational map $X \to E$.

The argument applies to any cubic surface in $\mathbb{A}^3$ containing two skew lines.

(11) Any irreducible closed set X is birational to a hypersurface of some affine space $\mathbb{A}^m$.