Basic Algebraic Geometry, Volume 1
Chapter 1, Section 2: Closed Subsets of Affine Space

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1. Definition of Closed Subsets

(1) A *closed subset* of $\mathbb{A}^n$ is a subset $X \subset \mathbb{A}^n$ consisting of all common zeros of a finite number of polynomials with coefficients in k. We will sometimes say simply closed set for brevity.

(2) A set defined by an infinite number of polynomials is also a closed set by the Hilbert Basis Theorem.

(3) The intersection of any number of closed sets is closed.

(4) The union of a finite number of closed sets is closed.

(5) A set $U \subset X$ is *open* if its complement $X \setminus U$ is closed. Any open set U containing x is called a *neighborhood* of x.

(6) The intersection of all the closed subsets of X containing a given subset $M \subset X$ is closed. It's called the *closure* of M and denoted $\overline{M}$.

(7) A subset M is *dense* in X if $\overline{M} = X$.

(8) The closed sets of $\mathbb{A}^1$ are the empty set, $\mathbb{A}^1$, and finite sets of points.

(9) The closed sets of $\mathbb{A}^2$ are the empty set, $\mathbb{A}^2$, finite sets of points, or the union of an algebraic plane curve and a finite set of points.

(10) If $X \subset \mathbb{A}^r$ and $Y \subset \mathbb{A}^s$ are closed, then $X \times Y \subset \mathbb{A}^{r+s}$ is a closed set.

(11) A set $X \subset \mathbb{A}^n$ defined by one equation is called a *hypersurface*.

2. Regular Functions on a Closed Subset

(1) Let $X \subset \mathbb{A}^n$ be a closed set. A function on X with values in k is *regular* if there exists a polynomial $F(T)$ with coefficients in k such that $f(x) = F(x)$ for all $x \in X$.

(2) The set of all polynomials vanishing on X is an ideal $\mathfrak{A}_X \subset k[T_1, \ldots, T_n]$.

(3) The set of regular functions on a closed set X forms an algebra $k[X]$ over k isomorphic to $k[T]/\mathfrak{A}_X$. So, $k[X]$ is determined by $\mathfrak{A}_X \subset k[T]$. The ring $k[X]$ is the *coordinate ring* of X.

(4) $k[X \times Y] = k[X] \otimes_k k[Y]$.

(5) $k[X]$ is a Noetherian ring.

(6) If $f \in k[X]$ is zero at every point $x \in X$ defined by the functions $g_1, \ldots, g_m$, then $f^r \in (g_1, \ldots, g_m)$ for some $r > 0$.

(7) $\mathfrak{A}_X = \sqrt{(g_1, \ldots, g_m)}$

(8) Closed subsets $X \subset Y$ iff $\mathfrak{A}_X \supset \mathfrak{A}_Y$.

(9) Each closed subset $X \subset Y$ corresponds to an ideal $\mathfrak{a}$ in $k[Y]$.

(10) Y is the empty set iff $\mathfrak{a}_{\mathrm{Y}} = k[X]$. Alternately, $\mathfrak{a}_{\mathrm{Y}}$ is the set of $f \in k[X]$ that vanish at all points of the subset Y.

(11) Each point $x \in X$ defines a maximal ideal $\mathfrak{m}_x \subset k[X]$ and conversely.

(12) For $f \in k[X]$, the set of points $x \in X$ at which f vanishes is denoted $V(f)$. It is a *hypersurface* in X.

3. Regular Maps

Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be closed subsets.

(1) A map $f : X \to Y$ is *regular* if there exists m regular functions $f_1, \ldots, f_m$ on X such that $f(x) = (f_1(x), \ldots, f_m(x))$ for all $x \in X$.

(2) Thus any regular map $f : X \to Y$ is given by $f(x) = (f_1(x), \ldots, f_m(x))$ if $G\left(f_1(x), \ldots, f_m(x)\right)$ for all $G \in \mathfrak{A}_Y$. That is, $f^* : k[Y] \to k[X]$.

(3)

Examples. (a) A regular function on X is the same as a regular map $X \to \mathbb{A}^1$.

(b) A linear map $\mathbb{A}^n \to \mathbb{A}^m$ is a regular map.

(c) The projection map $(x, y) \mapsto x$ defines a regular map of the curve $xy = 1$ to $\mathbb{A}^1$.

(d) Let $X \subset \mathbb{A}^n$ be a closed subset and F a regular function on X. Let $X' \subset X \times \mathbb{A}^1$ be defined by the equation $T_{n+1}F(T_1, \ldots, T_n) = 1$. The projection $\varphi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$ defines a regular map $\varphi : X' \to X$.

(e) The map $f(t) = (t^2, t^3)$ is a regular map of the line $\mathbb{A}^1$ to the curve given by $y^2 = x^3$.

(f) Suppose $X \subset \mathbb{A}^n$ has coefficients in $\mathbb{F}_p$. The map $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ defined by $\varphi(\alpha_1, \ldots, \alpha_n) = (\alpha_1^p, \ldots, \alpha_n^p)$. This is a regular map taking X to itself since $F_i(\alpha_1^p, \ldots, \alpha_n^p) = [F_i(\alpha_1, \ldots, \alpha_n)]^p$ in any field of characteristic p. The resulting map $\varphi : X \to X$ is the *Frobenius map*. The points of X fixed by φ are the points with coefficients in $\mathbb{F}_p$.

(4) A regular map $f : X \to Y$ of closed sets is an isomorphism if it has an inverse. That is, if there exists a regular map $g : Y \to X$ such that $f \circ g = 1$ and $g \circ f = 1$. In this case we say that X and Y are *isomorphic*.

(5) If $f : X \to Y$ is an isomorphism, then $f^* : k[Y] \to k[X]$ is an isomorphism of algebras. The converse is also true.

(6) **Theorem**: An algebra A over a field k is isomorphic to a coordinate ring $k[X]$ of some closed subset X if and only if A has no nilpotents and is finitely generated as an algebra over k.

(7) The functor $X \mapsto k[X]$ is an equivalence of categories between the category of closed subsets of affine space and regular maps with the category of reduced, finitely generated k-algebras and k-algebra homomorphisms.

(8)

Examples. (a) The subset $y = x^k$ is isomorphic to the line adn $f(x, y) = x$ and $g(t) = (t, t^k)$ define an isomorphism.

(b) The projection $f(x, y) = x$ of $xy = 1$ to the x-axis is not and isomorphism since it is not a one-to-one correspondence. (No point on $xy = 1$ goes onto $x = 0$.)

(c) The map $f(t) = (t^2, t^3)$ of the line to $y^2 = x^3$ is a one-to-one correspondence. However, it is not an isomorphism since the inverse map $g(x, y) = y/x$ and this function is not regular at 0.

(d) Let X and $Y \subset \mathbb{A}^n$ be closed sets. Consider $X \times Y \subset \mathbb{A}^{2n}$. Let $\Delta \subset \mathbb{A}^{2n}$ be defined by equations $t_1 = u_1, \ldots, t_n = u_n$, called the *diagonal*. Consider the map $X \cap Y \to \mathbb{A}^{2n}$ by $z \mapsto (z, z)$. This is an isomorphism of $X \cap Y$ with $(X \times Y) \cap \Delta$.

This allows us to reduce the study of the intersection of two closed subsets with the intersection of a different closed set and a linear subspace.

(e) Let X be a closed set and G a finite group of automorphisms of X. Suppose the characteristic of the field k does not divide the order N of G. Set $A = k[X]$, and let A^G be the subalgebra of invariants of G in A. This is a finitely generated, reduced algebra over k, so $A^G = k[Y]$ for some closed subset Y and a regular map $\varphi : X \to Y$ such that $\varphi^*(k[Y]) = A^G$. This set Y is called the *quotient variety* or *quotient space* of X by the action of G, and is written X/G.

If X/G parametrises the orbits $\{g(x) \mid g \in G\}$ of G acting on X.

(9) A homomorphism $f^* : k[Y] \to k[X]$ is injective if the image of $f : X \to Y$ is dense in Y. This is certainly true if $f(X) = Y$, but $f(X) \neq Y$ is also possible.

(10) Closed subsets of affine space have a geometry with a special flavor which is quite often nontrivial.

Theorem (Abhyankar and Moh)**.** *A curve $X \subset \mathbb{A}^2$ is isomorphic to $\mathbb{A}^1$ if and only if there exists an automorphism of $\mathbb{A}^2$ that takes XX to a line. (Here an automorphism is an isomorphism from $\mathbb{A}^2$ to itself.)*

(11) The group $\mathrm{Aut}(\mathbb{A}^2)$ is generated by maps of the form $x' = \alpha x$, $y' = \beta y + f(x)$ for $\alpha, \beta \neq 0$ in k and f a polynomial. $\mathrm{Aut}(\mathbb{A}^2)$ is the free product of two subgroups, (1) the maps of the forms above and (2) the affine maps over their common subgroup.