Chapter 1

Rings and Ideals

Exercises

Exercise 1. Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Solution. *Proof.* Let x be a nilpotent element and let u be a unit in a ring A.

Assume $u+x$ is not a unit of A. Since $u+x$ is not a unit of A, $u+x$ must be contained in some maximal ideal $\mathfrak{m}$. Since x is nilpotent, it is contained in every prime ideal and hence every maximal ideal, so $x \in \mathfrak{m}$. It follows that $u = (u+x) - x \in \mathfrak{m}$, which is absurd. So $u+x$ is a unit of A. $\square$

Exercise 2. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

i) f is a unit in $A[x] \Leftrightarrow a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

ii) f is nilpotent $\Leftrightarrow a_0, a_1, \ldots, a_n$ are nilpotent.

iii) f is a zero-divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$.

iv) f is said to be *primitive* if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\Leftrightarrow f$ and g are primitive.

Solution. i) *Proof.* ($\Leftarrow$) Suppose $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$, where a_0 is a unit and $a_1, a_2, \ldots, a_n$ are nilpotent. Since $a_1, a_2, \ldots, a_n$ are nilpotent, $a_1x, a_2x^2, \ldots, a_nx^n$ are all nilpotent, so their sum

$$a_1x + a_2x^2 + \cdots + a_nx^n$$

is also nilpotent. Since a_0 is a unit, $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a unit by Exercise 1. $\square$

($\Rightarrow$) Suppose $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$ is a unit. Let $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$ be the inverse of f. Since $fg = 1$, we must have $a_0b_0 = 1$, so a_0 is a unit in A.

The coefficient of x^{m+n} in the product is $a_nb_m = 0$. The coefficient of x^{m+n-1} in the product is $a_nb_{m-1} + a_{n-1}b_m = 0$. Multiplying this equation by a_n yields

$$a_n(a_nb_{m-1} + a_{n-1}b_m) = a_n^2b_{m-1} + a_{n-1}(a_nb_m) = a_n^2b_{m-1} = 0,$$

since $a_nb_m = 0$. Assume that $a_n^{r+1}b_{m-r} = 0$ for all $r \leq R < m$. The coefficient of $x^{m+n-R-1}$ is

$$a_nb_{m-R-1} + a_{n-1}b_{m-R} + a_{n-2}b_{m-R+1} + \cdots + a_{n-R-1}b_m = 0$$

Multiplying this equation by a_n^{R+1} yields

$$a_n^{R+2}b_{m-R-1} + a_{n-1}(a_n^{R+1}b_{m-R}) + a_{n-2}a_n(a_n^Rb_{m-R+1}) + \cdots + a_{n-R-1}a_n^R(a_nb_m) = 0$$

By the inductive hypothesis, all the terms in parentheses are zero, so we have

$$a_n^{R+2}b_{m-R-1} = 0.$$

We conclude that $a_n^{r+1}b_{m-r} = 0$ for all for all $r \leq m$. In particular, we have $a_n^{m+1}b_0 = 0$. Since b_0 is a unit, we must have $a_n^{m+1} = 0$. That is, a_n is nilpotent.

Since a_n is nilpotent, a_nx^n is nilpotent. Using Exercise 1, we have that $f - a_nx^n = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ is a unit. By induction, a_1, a_2, a_3, $\ldots$, a_n are all nilpotent, as desired.

ii) ($\Rightarrow$) Suppose $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$ is nilpotent. Then xf is likewise nilpotent, so $1 + xf$ is a unit by Exercise 1. Then, by part (i), we must have a_0, a_1, $\ldots$, a_n are all nilpotent.

($\Leftarrow$) Suppose $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$ with a_0, $\ldots$, a_n all nilpotent. Then a_1x, a_2x^2,$\ldots$, a_nx^n are all nilpotent, and therefore $f = a_0 + a_1x + \cdots + a_nx^n$ is likewise nilpotent.

iii) If there exists $a \neq 0$ in A such that $af = 0$, then f is a zero divisor by definition, so there's nothing to prove.

Suppose $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$ is a zero divisor. If f is zero, there's nothing to prove, so assume $f \neq 0$.

Let $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m \neq 0$ be of minimal degree so that $fg = 0$. The coefficient of the $n + m$ term is a_nb_m which must be zero. Then a_ng has degree less than m and $(a_ng)f = 0$, so $a_ng \equiv 0$ by minimality. So, $a_nb_j = 0$ for all $0 \leq j \leq m$.

Next, consider the coefficient of the $n+m-1$ term is $a_nb_{m-1}+a_{n-1}b_m = a_{n-1}b_m$ which must be zero. Then $a_{n-1}g$ has degree less than m and $(a_{n-1}g)f = 0$, so $a_{n-1}g \equiv 0$ by minimality. So, $a_{n-1}b_j = 0$ for all $0 \leq j \leq m$.

Suppose $a_{n-i}b_j = 0$ for all $0 \leq j \leq m$ and for all $0 \leq i < r \leq n$. We extend the definitions of b_ℓ to be zero if ℓ is not between 0 and m. Then the coefficient of the $n - r + m$ term is

$$a_nb_{m-r} + a_{n-1}b_{m-r+1} + \cdots + a_{n-r+1}b_{m-1} + a_{n-r}b_m = a_{n-r}b_m,$$

which must be zero. Then $a_{n-r}g$ has degree less than m and $(a_{n-r}g)f = 0$, so $a_{n-r}g \equiv 0$ by minimality. So, $a_{n-r}b_j = 0$ for all $0 \leq j \leq m$.

By induction, it follows that $a_ib_j = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$. Let a be any $b_j \neq 0$. Then $af = 0$.

iv) Let A be a UFD. We first show that the product of two primitive polynomials f and g is primitive.

Suppose p is prime and divides all the coefficients of fg. Since f and g are primitive, there exist coefficients of f and coefficients of g that are not divisible by p. Let a_rx^r and b_sx^s be the lowest degree terms with a coefficients not divisible by p in f and g, respectively. The coefficient of x^{r+s} in fg is then $\sum_{i=0}^{r+s} a_ib_{r+s-i}$. Now, for $i < r$, p divides a_i and for $i > r$, p divides b_{r+s-i}. It follows that p divides a_rb_s. Since p is irreducible and A is a UFD, it follows that p divides a_r or p divides b_s, either of which is a contradiction. It follows that fg is primitive.

Conversely, suppose that fg is primitive. Let k be the greatest common divisor of the coefficients of f. Let ℓ be the greatest common divisor of the coefficients of g. Let $f' = f/k$ and $g' = g/\ell$. Then f' and g' are primitive, so by the last paragraph, $f'g'$ is primitive. We have that $f = kf'$ and $g = \ell g'$, so that $fg = (k\ell)f'g'$. Since $f'g'$ is primitive, the greatest common divisor of the coefficients of fg is $k\ell$, so we have $k\ell$ is a unit. That is, k and ℓ are units, so f and g are primitive.

Exercise 3. Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \ldots, x_n]$ in several indeterminates.

Solution. Let A be a ring and let $A[x_1, \ldots, x_n]$ be the ring of polynomials in n indeterminates with coefficients in A. Let $f = \sum_{I=(d_0,\ldots,d_n)} a_I x_1^{d_1} \cdots x_n^{d_n} \in A[x_1, \ldots, x_n]$. Prove that

i) f is a unit in $A[x_1, \ldots, x_n] \Leftrightarrow$ the constant coefficient is a unit and each a_I is nilpotent for $|I| > 0$.

ii) f is nilpotent $\Leftrightarrow$ each a_I is nilpotent.

iii) f is a zero-divisor $\Leftrightarrow bf = 0$ for some $b \in A$, $b \neq 0$.

iv) f is said to be *primitive* if $\gcd(a_I) = (1)$. Prove that if $f, g \in A[x_1, \ldots, x_n]$, then fg is primitive $\Leftrightarrow f$ and g are primitive.

Proof. We can write

$$f = \sum_{i=1}^{n} g_i x_n^i \quad \text{where } g_i \in A[x_1, \ldots, x_{n-1}].$$

ii) ($\Rightarrow$) Suppose f is nilpotent in $A[x_1, \ldots, x_n]$. By Exercise 2, each g_i is nilpotent. By induction, it follows that all the coefficients of the g_i's are nilpotent. So, a_I is nilpotent for all I.

($\Leftarrow$) Suppose each a_I is nilpotent. Since each a_I is nilpotent, each $a_I x^I$ is nilpotent, and since the nilradical is an ideal, the sum of these elements is also nilpotent.

i) ($\Rightarrow$) Suppose f is a unit. Writing f as $f = \sum_{i=1}^{n} g_i x_n^i$ with $g_i \in A[x_1, \ldots, x_{n-1}]$, as in Exercise 2(i), we must have g_0 is a unit and g_i is nilpotent for $1 \leq i \leq n$. By induction on the degree applied to g_0, the constant term in g_0—which is also the constant term in f—is a unit and the remaining coefficients of g_0 are nilpotent. Since g_i is nilpotent for $1 \leq i \leq n$, by part (ii) of this problem, all the coefficients of g_i are nilpotent. So, we see that in f, the constant term is a unit and the remaining coefficients are nilpotent.

($\Leftarrow$) Suppose f has a unit for a constant term and nilpotent elements for the remaining coefficients. Then f is the sum of a unit and a nilpotent element by part (ii). It follows that f is a unit by Exercise 1.

iii) ($\Rightarrow$) Suppose f is a zero-divisor. Let $g \in A[x_1, \ldots, x_n]$ be of minimal degree so that $fg \equiv 0$.

Mark, finish this one.

($\Leftarrow$) Suppose there exists $b \in A$, $b \neq 0$, so that $bf = 0$. Then f is a zero-divisor by definition.

iv) Mark, do this one.

($\Rightarrow$)

($\Leftarrow$)

□

Exercise 4. In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Solution. *Proof.* Let $\mathfrak{N}$ be the nilradical of A and let $\mathfrak{R}$ be the Jacobson radical of A. Since every maximal ideal is a prime ideal, we always have $\mathfrak{N} \subseteq \mathfrak{R}$.

Let $f \in \mathfrak{R}$. Then $1 + fx$ is a unit, by Proposition 1.9. By Exercise 2(i), we have that all the coefficients of f are nilpotent, whereby f is nilpotent, by Exercise 2(ii). Hence $f \in \mathfrak{N}$, as desired. □

Exercise 5. Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A. Show that

i) f is a unit in $A[[x]] \Leftrightarrow a_0$ is a unit in A.

ii) If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true? (See Chapter 7, Exercise 2.)

iii) f belongs to the Jacobson radical of $A[[x]] \Leftrightarrow a_0$ belongs to the Jacobson radical of A.

iv) The contraction of a maximal ideal m of $A[[x]]$ is a maximal ideal of A, and m is generated by m^c and x.

v) Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.

Solution. *Proof.* Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A.

i) $(\Rightarrow)$ Suppose $f \in A[[x]]$ is a unit. Let $g = \sum_{n=0}^{\infty} b_n x^n \in A[[x]]$ be such that $fg = 1$. As in the proof of Exercise 2(i), $a_0 b_0 = 1$, so a_0 is a unit.

   $(\Leftarrow)$ Suppose a_0 is a unit in A. We will construct an inverse g for f by constructing its coefficients inductively. Let $g = \sum_{n=0}^{\infty} b_n x^n \in A[[x]]$ be the inverse of f to be constructed. Since $fg = 1$, we must have $a_0 b_0 = 1$, so $b_0 = a_0^{-1}$.

   Set $b_1 = -a_0^{-1} a_1 b_0$. Then the coefficient of x in the product fg is

$$\begin{aligned} a_1 b_0 + a_0 b_1 &= a_1 b_0 + a_0(-a_0^{-1} a_1 b_0) \\ &= a_1 b_0 + (a_0 \cdot -a_0^{-1}) a_1 b_0 \\ &= a_1 b_0 - a_1 b_0 \\ &= 0. \end{aligned}$$

Suppose the coefficient b_n has been constructed so that the coefficient of x_n in the product fg is zero. That is

$$a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0 = 0$$

Let

$$b_{n+1} = -a_0^{-1}(a_1b_n + \cdots + a_nb_1 + a_{n+1}b_0)$$

Then the coefficient of x^{n+1} is

$$\begin{aligned} a_0b_{n+1} + a_1b_n + \cdots + a_nb_1 + a_{n+1}b_0 &= a_0 \cdot -a_0^{-1}(a_1b_n + \cdots + a_nb_1 + a_{n+1}b_0) \\ &\qquad \cdots + a_1b_n + \cdots + a_nb_1 + a_{n+1}b_0 \\ &= -(a_1b_n + \cdots + a_nb_1 + a_{n+1}b_0) \\ &\qquad \cdots + a_1b_n + \cdots + a_nb_1 + a_{n+1}b_0 \\ &= 0. \end{aligned}$$

By induction, b_n can be constructed for all n so that the coefficient of x_n is zero for all $n \geq 1$. Hence, f is a unit having inverse

$$g = \sum_{n=0}^{\infty} b_nx^n,$$

as required.

ii) ($\Rightarrow$) Suppose $f \in A[[x]]$ is nilpotent. Then $f^n = 0$ for some $n \geq 0$ and since the constant term here is a_0^n, we must have $a_0^n = 0$, whereby a_0 is nilpotent. Since f is nilpotent and a_0 is nilpotent, so is $f - a_0 = \sum_{n=1}^{\infty} a_nx^n = x\sum_{n=1}^{\infty} a_nx^{n-1} = x\sum_{n=0}^{\infty} a_{n+1}x^n$. This means $\sum_{n=0}^{\infty} a_{n+1}x^n$ is nilpotent. Since $\sum_{n=0}^{\infty} a_{n+1}x^n$ is nilpotent, the constant term a_1 must likewise be nilpotent. Continuing in this fashion, a_n must be nilpotent for all n.

Mark, find the counterexample.

iii) ($\Rightarrow$) Suppose f belongs to the Jacobson radical of $A[[x]]$. By Proposition 1.1.9, for any $y \in A$, $1 - yf$ is a unit in $A[[x]]$. By part (i), since $1 - yf$ is a unit in $A[[x]]$, the constant term of $1 - yf$ is a unit in A, and this occurs for all $y \in A$. But the constant term of $1 - yf$ is $1 - ya_0$, so we have $1 - ya_0$ is a unit in A for all $y \in A$. By Proposition 1.1.9 again, a_0 lies in the Jacobson radical of A.

($\Leftarrow$) Suppose a_0 lies in the Jacobson radical of A. Let $g = \sum_j b_jx^j$ be any element in $A[[x]]$. The constant term of $1 - fg$ is $1 - a_0b_0$. Since a_0 lies in the Jacobson radical of A, by Proposition 1.1.9, $1 - a_0b_0$ is then a unit in A. By part (i), $1 - fg$ is a unit in $A[[x]]$. Since $g \in A[[x]]$ is arbitrary, f lies in the Jacobson radical of $A[[x]]$, by Proposition 1.1.9 again.

iv) Let $\mathfrak{m}$ be a maximal ideal in $A[[x]]$ and consider its contraction $\mathfrak{m}^c = A \cap \mathfrak{m}$ in A. Since 0 lies in the Jacobson radical of A, x lies in the Jacobson radical of $A[[x]]$ by part (iii). Hence, $x \in \mathfrak{m}$. If $f = a + gx \in \mathfrak{m}$, then $a = f - gx \in \mathfrak{m}$ since both f and x lie in $\mathfrak{m}$. Hence $a \in \mathfrak{m}^c$. This shows $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.

The natural inclusion $A \hookrightarrow A[[x]]$ induces a homomorphism $A/\mathfrak{m}^c \to A[[x]]/\mathfrak{m}$ and this latter ring is a field since $\mathfrak{m}$ is maximal. I claim that $A/\mathfrak{m}^c$ is a subfield of this field, showing that $\mathfrak{m}^c$ is maximal in A.

Let $a \in A/\mathfrak{m}^c$ be nonzero. This element is also nonzero in $A[[x]]/\mathfrak{m}$, so it has an inverse in the field $A[[x]]/\mathfrak{m}$. Let $f + \mathfrak{m}$ be its inverse. Then $af + \mathfrak{m} = 1 + \mathfrak{m}$ whereby $1 - af \in \mathfrak{m}$. Write $f = a_0 + gx$ for some $a_0 \in A$ and $g \in A[[x]]$. Then $1 - af = 1 - a(a_0 + gx) = (1 - aa_0) - agx \in \mathfrak{m}$. Since $x \in \mathfrak{m}$, we have that $1 - aa_0$ lies both in $\mathfrak{m}$ and in A. So, $1 - aa_0 \in \mathfrak{m}^c$. Hence $(a + \mathfrak{m}^c)(a_0 + \mathfrak{m}^c) = 1 + \mathfrak{m}^c$. This shows $A/\mathfrak{m}^c$ is a field, so $\mathfrak{m}^c$ is a maximal ideal in A.

v) Let $\mathfrak{p}$ be a prime ideal in A. Let $\mathfrak{q}$ be the ideal consisting of all $\sum a_k x^k$ where $a_0 \in \mathfrak{p}$. This is, in fact, an ideal. If $fg \in \mathfrak{q}$ where $f = \sum_k a_k x^k$ and $g = \sum_k b_k x^k$, then $a_0 b_0 \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, either a_0 or b_0 lies in $\mathfrak{p}$. But this says $f \in \mathfrak{q}$ or $g \in \mathfrak{q}$, so $\mathfrak{q}$ is a prime ideal. It's easy to see that $\mathfrak{p} = \mathfrak{q}^c$.

□

Exercise 6. A ring A is such that every ideal not contained in the nilradical contains a non-zero idempotent (that is, an element e such that $e^2 = e \neq 0$.) Prove that the nilradical and Jacobson radical of A are equal.

Solution. *Proof.* Let A be a ring such that every ideal not contained in the nilradical contains a non-zero idempotent (that is, an element e such that $e^2 = e \neq 0$.) Let $\mathfrak{N}$ be the nilradical of A and let $\mathfrak{R}$ be the Jacobson radical of A. As always, $\mathfrak{N} \subseteq \mathfrak{R}$.

Assume the Jacobson radical is not contained in the nilradical. Since the Jacobson radical is not contained in the nilradical, there exists a nonzero idempotent $e \in \mathfrak{R}$, by hypothesis. Hence, $e^2 = e$, so $e(1 - e) = 0$. However, $e \in \mathfrak{R}$, so by Proposition 1.9, $1 - e$ is a unit, so $e = 0$, which is a contradiction. So, the nilradical equals the Jacobson radical. □

Exercise 7. Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Solution. *Proof.* Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Let $\mathfrak{p}$ be a prime ideal in A. Let $\overline{x} \in A/\mathfrak{p}$ be nonzero. By hypothesis, there exists $n > 1$ so that $x^n = x$, so in the quotient ring we have $\overline{x}^n = \overline{x}$. Since $A/\mathfrak{p}$ is an integral domain and $x \notin \mathfrak{p}$, we must have $\overline{x}^{n-1} = \overline{1}$. Hence $\overline{x}$ is a unit with inverse $\overline{x}^{n-2}$. Thus we see that every nonzero element $\overline{x}$ in $A/\mathfrak{p}$ is a unit, so $A/\mathfrak{p}$ is a field. It now follows that $\mathfrak{p}$ is maximal ideal. Since $\mathfrak{p}$ is an arbitrary prime ideal, we see that every prime ideal in A is maximal. $\square$

Exercise 8. Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Solution. *Proof.* Order the set of prime ideals in A by inclusion. We now apply Zorn's lemma. Let

$$\mathfrak{p}_1 \supseteq \mathfrak{p}_2 \supseteq \mathfrak{p}_3 \supseteq \mathfrak{p}_4 \supseteq \cdots$$

be a chain of prime ideals in A.

Let $\mathfrak{p} = \cap_{n=1}^{\infty} \mathfrak{p}_n$. Suppose $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$. Then there exist n and m so that $x \notin \mathfrak{p}_n$ and $y \notin \mathfrak{p}_m$. Let $k = \max\{m, n\}$. Since $x \notin \mathfrak{p}_n \supseteq \mathfrak{p}_k$ and since $y \notin \mathfrak{p}_m \supseteq \mathfrak{p}_k$. Since $\mathfrak{p}_k$ is a prime ideal, $xy \notin \mathfrak{p}_k$, so that $xy \notin \mathfrak{p}$. This shows $\mathfrak{p}$ is a prime ideal. By Zorn's lemma, the set of prime ideals of A has minimal elements with respect to inclusion. $\square$

Exercise 9. Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = \operatorname{rad}(\mathfrak{a}) \Leftrightarrow \mathfrak{a}$ is an intersection of prime ideals.

Solution. *Proof.* ($\Rightarrow$) Suppose $\mathfrak{a} = \operatorname{rad}(\mathfrak{a})$. Since $\operatorname{rad}(\mathfrak{a}) = \mathfrak{N}_{A/\mathfrak{a}}$, $\mathfrak{a}$ is an intersection of prime ideals by Proposition 1.1.8.

($\Leftarrow$) Suppose that an ideal $\mathfrak{a}$ is an intersection of prime ideals. Since each of these prime ideals contains $\mathfrak{a}$, the set of all prime ideals containing $\mathfrak{a}$ includes these prime ideals. Since the intersection of all prime ideals containing $\mathfrak{a}$ is $\operatorname{rad}(\mathfrak{a})$, we have $\operatorname{rad}(\mathfrak{a}) \subseteq \mathfrak{a}$. Since the reverse inclusion is clear, we have $\mathfrak{a} = \operatorname{rad}(\mathfrak{a})$. $\square$

Exercise 10. Let A be a ring, $\mathfrak{N}$ its nilradical. Show that the following are equivalent:

i) A has exactly one prime ideal;

ii) every element of A is either a unit or nilpotent;

iii) $A/\mathfrak{N}$ is a field.

Solution. *Proof.* i) $\Rightarrow$ ii) Suppose A contains exactly one prime ideal $\mathfrak{p}$. Let $x \in A$ be arbitrary. Suppose $x \in \mathfrak{p}$. Since the nilradical of A is $\mathfrak{p}$, we have that x is nilpotent. Suppose that $x \notin \mathfrak{p}$. Since every nonunit is contained in a maximal ideal and $\mathfrak{p}$ is the only prime ideal and therefore the only maximal ideal, x must be a unit. Hence, every element of A is either nilpotent or a unit.

ii) $\Rightarrow$ iii) Suppose every element of A is either a unit or nilpotent. Let $\overline{x} \in A/\mathfrak{N}$, $\overline{x} \neq \overline{0}$. Since $\overline{x} \neq \overline{0}$, $x \notin \mathfrak{N}$, so, by hypothesis, x is a unit. Hence, there exists $y \in A$ with $xy = 1$. Then $\overline{x} \cdot \overline{y} = \overline{1}$, whereby $\overline{x}$ is a unit in $A/\mathfrak{N}$. Since $\overline{x}$ is an arbitrary nonzero element of $A/\mathfrak{N}$, we conclude that $A/\mathfrak{N}$ is a field.

iii) $\Rightarrow$ i) Suppose $A/\mathfrak{N}$ is a field. Since there is a one-to-one correspondence between ideals in $A/\mathfrak{N}$ and ideals in A which contain $\mathfrak{N}$, and since any field only contains two trivial ideals, we see that there is no proper ideal which properly contains $\mathfrak{N}$. That is, $\mathfrak{N}$ is maximal and so must be the only prime ideal in A.

□

Exercise 11. A ring A is *Boolean* if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that

i) $2x = 0$ for all $x \in A$;

ii) every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements;

iii) every finitely generated ideal in A is principal.

Solution. *Proof.* i) Let $x \in A$. Then

$$\begin{aligned}(x+1)^2 &= x+1\\ x^2+2x+1 &= x+1\\ x+2x+1 &= x+1\\ 2x &= 0,\end{aligned}$$

since $x^2 = x$ for all $x \in A$.

ii) Let $\mathfrak{p}$ be a prime ideal in A. By Exercise 7, $\mathfrak{p}$ is maximal.

Let $\overline{x} \in A/\mathfrak{p}$ be represented by $x \in A$. Since $x^2 = x$ in A, $\overline{x}^2 = \overline{x}$ in $A/\mathfrak{p}$, and since $A/\mathfrak{p}$ is a field, either $\overline{x} = \overline{0}$ or $\overline{1}$. So, we see that $A/\mathfrak{p}$ is the field with two elements.

iii) Let $\mathfrak{a}$ be a finitely generated ideal. Let's say $\mathfrak{a}$ is generated by two elements, x and y. Then

$$\mathfrak{a} = \{ax + by \,|\, a, b \in A\}$$

Let $z = x + y + xy$. Then

$$\begin{aligned} xz &= x(x + y + xy) = x^2 + xy + x^2 y = x \\ yz &= y(x + y + xy) = xy + y^2 + xy^2 = y. \end{aligned}$$

So, $(x, y) = (z)$. The result follows by induction on the number of generators.

□

Exercise 12. A local ring contains no idempotent unequal to zero or one.

Solution. *Proof.* Let R be a local ring and let x be any idempotent.

Suppose x is a unit and let y be the multiplicative inverse of x. Then

$$\begin{aligned} x^2 &= x \\ x^2 y &= xy \\ x(xy) &= xy \\ x &= 1. \end{aligned}$$

On the other hand, suppose x is not a unit. Let $\mathfrak{p}$ be a prime ideal in R. Considering the integral domain $R/\mathfrak{p}$, we have $\overline{x}^2 - \overline{x} = \overline{x}(\overline{x} - 1) = 0$ in $R/\mathfrak{p}$, so either $\overline{x} = 0$ or $\overline{x} = 1$ in $R/\mathfrak{p}$.

Suppose $\overline{x} = 1$ in $R/\mathfrak{p}$. Then $x = 1 + y$ for some $y \in \mathfrak{p}$. Now x and y are necessarily nonunits, so they must be contained in the unique maximal ideal of R. Then $1 = x - y$ is also in this maximal ideal, a contradiction. So, we must have $\overline{x} = 0$ in $R/\mathfrak{p}$. That is, $x \in \mathfrak{p}$.

Since $\mathfrak{p}$ is an arbitrary prime ideal in R, it follows that x is contained in every prime ideal in R, so x is nilpotent. However, since $x^2 = x$, by induction $x^n = x$ for all $n \in \mathbb{N}$, and since x is nilpotent, $x = 0$.

This concludes the proof. □

Exercise 13. Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K. Let A be the polynomial ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{a}$, and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K, obtaining a field K_2, and so on. Let $L = \bigcup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let $\overline{K}$ be the set of all elements of L which are algebraic over K. Then $\overline{K}$ is an algebraic closure of K.

Solution. *Proof.* Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K. Let A be the polynomial ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$.

Suppose $\mathfrak{a} = A$. Then there is an n and polynomials $g_1, \ldots, g_n$ such that

$$1 = f_1(x_{i_1})g_1(x_{i_1}, \ldots, x_{i_n}) + \cdots + f_n(x_{i_n})g_n(x_{i_1}, \ldots, x_{i_n})$$

For simplicity we shall write x_m in place of x_{i_m} for each m. We can assume that all the g_m involve only the variables, $x_1, \ldots, x_n$ by increasing the number of f_m if necessary in an equation of this type.

Suppose n is chosen to be minimal such that we have such an expression involving n of the x_i. Let $S = K[x_1, \ldots, x_n]$, so that then $(f(x_1), \ldots, f_n(x_n)) = S$. Let $R = K[x_1, \ldots, x_{n-1}]$. By minimality of n we have $(f(x_1), \ldots, f_{n-1}(x_{n-1})) \neq R$. Let us view the above equation as taking place in $S = R[x_n]$. If $c_m = f_m(x_m) \in R$ we have $J = (c_1, \ldots, c_{n-1}, f_n(x_n)) = S$. Set $I_0 = (c_1, \ldots, c_{n-1}) \subseteq R$ so that $J = (I_0, f_n(x_n))$. There are ring homomorphisms

$$R[x_n] \to (R/I_0)[x_n] \to \frac{(R/I_0)[x_n]}{(\overline{f_n(x_n)})}$$

where $\overline{f_n(x_n)}$ is the image of $f_n(x_n)$ in $(R/I_0)[x_n]$. Since R/I_0 is a nonzero ring and $\overline{f_n(x_n)}$ is not a unit (as f_n is monic of degree at least 1) we see that this last ring is nonzero. Hence the kernel of the composite homomorphism is a proper ideal of S. But J lies in this kernel, so $J \neq S$. This contradiction shows our original I is a proper ideal of $K[\{x_i\}]$, finishing the first part of the proof.

Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{a}$, and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K, obtaining a field K_2, and so on. Let $L = \bigcup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let $\overline{K}$ be the set of all elements of L which are algebraic over K. We claim that $\overline{K}$ is an algebraic closure of K.

Let $f \in K[x]$ be irreducible. Then f is a constant multiple of one of the monic irreducible polynomials in Σ, so in K_1 we have a root x_f for f. The result follows by induction on the degree of f. $\square$

Exercise 14. In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

Solution. *Proof.* Let A be a ring and let Σ be the set of all ideals in which every element is a zero-divisor. The set Σ is nonempty since $(0) \in \Sigma$.

Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ be a chain of ideals in Σ. Let $\mathfrak{a} = \bigcup_i \mathfrak{a}_i$. As in the proof of Theorem 1.1.3, $\mathfrak{a}$ is an ideal and since all the ideals consist only of elements which are zero-divisors, we see that $\mathfrak{a}$ contains only zero-divisors, so $\mathfrak{a} \in \Sigma$. By Zorn's lemma, Σ contains a maximal element.

Let $\mathfrak{p}$ be a maximal element in Σ. Let x and y be elements of A which are not in $\mathfrak{p}$. By the maximality of $\mathfrak{p}$, $\mathfrak{p} + (x)$ is not in Σ, so it contains an element u which is not a zero divisor. Similarly, $\mathfrak{p} + (y)$ is not in Σ, so it contains an element v which is not a zero divisor. Since u and v are not zero-divisors, neither is $uv \in \mathfrak{p} + (xy)$. It follows that $xy \notin \mathfrak{p}$, so $\mathfrak{p}$ is a prime ideal.

If x is a zero-divisor, the principal ideal (x) is in Σ and since Σ contains maximal elements which are prime, x lies in some prime ideal consisting only of zero-divisors. It follows that the set of zero-divisors in A is a union of prime ideals. □

Exercise 15. Let A be a ring and let X be the set of all prime ideals of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

ii) $V(0) = X$, $V(1) = \emptyset$.

iii) if $(E_i)_{i \in I}$ is any family of subsets of A, then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}$, $\mathfrak{b}$ of A.

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology.* The topological space X is called the *prime spectrum* of A, and is written $\operatorname{Spec}(A)$.

Solution. *Proof.* Let A be a ring and let X be the set of all prime ideals of A, let $V(E)$ denote the set of all prime ideals of A which contain E.

i) Let $\mathfrak{a}$ be the ideal generated by E. Then $E \subseteq \mathfrak{a} \subseteq r(\mathfrak{a})$, so

$$V(r(\mathfrak{a})) \subseteq V(\mathfrak{a}) \subseteq V(E).$$

On the other hand, let $\mathfrak{p}$ be a prime ideal in $V(E)$. Then $\mathfrak{p}$ contains E, and since $\mathfrak{a}$ is the smallest ideal containing E, we have $\mathfrak{a} \subseteq \mathfrak{p}$. Let $x \in \mathrm{rad}\,(\mathfrak{a})$. Then $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}$, so that $x^n \in \mathfrak{a} \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, $x \in \mathfrak{p}$. Since $x \in \mathrm{rad}\,(\mathfrak{a})$ is arbitrary, $\mathrm{rad}\,(\mathfrak{a}) \subseteq \mathfrak{p}$. So, $\mathfrak{p} \in V(\mathrm{rad}\,(\mathfrak{a}))$. Since $\mathfrak{p} \in V(E)$ is arbitrary, $V(E) \subseteq V(\mathrm{rad}\,(\mathfrak{a}))$.

From the inclusions above, this implies that

$$V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a})).$$

This concludes the proof of part (i).

ii) It is clear that $V(0) = X$ since every prime ideal contains the zero ideal. Similarly, it is clear that $V(1) = \emptyset$ since no prime ideal contains 1.

iii) Let $(E_i)_{i\in I}$ be any family of subsets of A. Let $\mathfrak{p} \in V(\bigcup_{i\in I} E_i)$. Then $\mathfrak{p} \supseteq \bigcup_{i\in I} E_i \supseteq E_i$ for each $i \in I$. Hence, $\mathfrak{p} \in V(E_i)$ for all $i \in I$, whereby $\mathfrak{p} \in \bigcap_{i\in I} V(E_i)$. Since $\mathfrak{p} \in V(\bigcup_{i\in I} E_i)$ is arbitrary, $V(\bigcup_{i\in I} E_i) \subseteq \bigcap_{i\in I} V(E_i)$.

On the other hand, let $\mathfrak{p} \in \bigcap_{i\in I} V(E_i)$. Then $\mathfrak{p} \in V(E_i)$ for all $i \in I$, whereby $\mathfrak{p} \supseteq E_i$ for all $i \in I$. Hence $\mathfrak{p} \supseteq \bigcup_{i\in I} E_i$. So, $\mathfrak{p} \in V(\bigcup_{i\in I} E_i)$. Since $\mathfrak{p} \in \bigcap_{i\in I} V(E_i)$ is arbitrary, $\bigcap_{i\in I} V(E_i) \subseteq V(\cup_{i\in I} E_i)$.

Putting the two inclusions together, we see that

$$\bigcap_{i\in I} V(E_i) = V\left(\bigcup_{i\in I} E_i\right),$$

as desired.

iv) Since $\mathfrak{a} \supseteq \mathfrak{a} \cap \mathfrak{b} \supseteq \mathfrak{ab}$, we have $V(\mathfrak{a}) \subseteq V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{ab})$. Similarly, since $\mathfrak{b} \supseteq \mathfrak{a} \cap \mathfrak{b} \supseteq \mathfrak{ab}$, we have $V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{ab})$. Hence

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{ab}).$$

Now, let $\mathfrak{p} \in V(\mathfrak{ab})$. Then $\mathfrak{p} \supseteq \mathfrak{ab}$, and since $\mathfrak{p}$ is a prime ideal, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. So, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, whereby $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. Since $\mathfrak{p} \in V(\mathfrak{ab})$ is arbitrary, we have $V(\mathfrak{ab}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.

Putting these two inclusions together yields

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{ab}) = V(\mathfrak{a} \cap \mathfrak{b}).$$

$\square$

Exercise 16. Draw pictures of $\mathrm{Spec}(\mathbb{Z})$, $\mathrm{Spec}(\mathbb{R})$, $\mathrm{Spec}(\mathbb{C}[x])$, $\mathrm{Spec}(\mathbb{R}[x])$, $\mathrm{Spec}(\mathbb{Z}[x])$.

Solution. $\mathrm{Spec}(\mathbb{Z})$ has two types of points: (0), which is dense, and (p), where $p \in \mathbb{Z}$ is prime. This ideal is maximal, so (p) is a closed point in $\mathrm{Spec}(\mathbb{Z})$.

$\mathrm{Spec}(\mathbb{R})$ only has one point, (0), which is closed.

$\mathrm{Spec}(\mathbb{C}[x])$ has two types of points: (0), which is dense, and $(f(x))$, where $f(x) \in \mathbb{C}[x]$ is irreducible. This ideal is maximal, so $(f(x))$ is a closed point in $\mathrm{Spec}(\mathbb{C}[x])$. Since the only irreducible polynomials in $\mathbb{C}[x]$ are the linear ones, this gives a one-to-one correspondence between the closed points of $\mathrm{Spec}(\mathbb{C}[x])$ and $\mathbb{C}$ itself.

$\mathrm{Spec}(\mathbb{R}[x])$ has two types of points: (0), which is dense, and $(f(x))$, where $f(x) \in \mathbb{R}[x]$ is irreducible. This ideal is maximal, so $(f(x))$ is a closed point in $\mathrm{Spec}(\mathbb{R}[x])$. The only irreducible polynomials in $\mathbb{R}[x]$ are either the linear ones or the irreducible quadratic ones. The points in $\mathrm{Spec}(\mathbb{R}[x])$ represented by linear polynomials give a one-to-one correspondence between those points and elements of $\mathbb{R}$. The points in $\mathrm{Spec}(\mathbb{R}[x])$ represented by irreducible quadratic polynomials give a one-to-one correspondence between those points and conjugate pairs of non-real complex numbers in $\mathbb{C}$.

$\mathrm{Spec}(\mathbb{Z}[x])$ has four types of points:

i) (0), which is dense

ii) $(f(x))$, where $f(x) \in \mathbb{Z}[x]$ is irreducible

iii) (p), where $p \in \mathbb{Z}$ is prime

iv) $(p, f(x))$, where $f(x)$ is irreducible modulo p. These ideals are maximal, so the points they represent are closed.

Exercise 17. For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \mathrm{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

i) $X_f \cap X_g = X_{fg}$;

ii) $X_f = \emptyset \iff f$ is nilpotent;

iii) $X_f = X \iff f$ is a unit;

iv) $X_f = X_g \iff \mathrm{rad}((f)) = \mathrm{rad}((g))$;

v) X is quasi-compact (that is, every open covering of X has a finite subcovering).

vi) More generally, each X_f is quasi-compact.

vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f.

The sets X_f are called *basic open sets* of $X = \text{Spec}(A)$.

Solution. *Proof.* For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \text{Spec}(A)$.

Let $U \subseteq \text{Spec}(A)$ be open and let $\mathfrak{p} \in U$. Since U is open, $U = \text{Spec}(A) \setminus V(\mathfrak{q})$ for some prime ideal $\mathfrak{q} \subseteq A$. Since $\mathfrak{p} \in U$, $\mathfrak{p} \not\supseteq \mathfrak{q}$, so there exists $f \in \mathfrak{q}$, $f \notin \mathfrak{p}$. Then $\mathfrak{p} \in X_f$ and $X_f \cap V(\mathfrak{q}) = \emptyset$ whereby $X_f \subseteq U$. This shows that basic open sets X_f form a basis for the Zariski topology on $\text{Spec}(A)$.

i) Since $(fg) \subseteq (f)$, we have $V(f) \subseteq V(fg)$, whereby $X_{fg} \subseteq X_f$. Similarly, since $(fg) \subseteq (g)$, we have $V(g) \subseteq V(fg)$, whereby $X_{fg} \subseteq X_g$. Hence $X_{fg} \subseteq X_f \cap X_g$. On the other hand, let $\mathfrak{p} \in X_f \cap X_g$. Then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, $fg \notin \mathfrak{p}$, so $\mathfrak{p} \in X_{fg}$. Hence $X_f \cap X_g \subseteq X_{fg}$. Putting these two inclusions together, we have
$$X_{fg} = X_f \cap X_g.$$

ii) Suppose $X_f = \emptyset$. Then $V(f) = \text{Spec}(A)$, which means that f is contained in every prime ideal in A. Hence, f is nilpotent.

Conversely, suppose that f is nilpotent. Then f is contained in every prime ideal in A. Consequently, $X_f = \emptyset$.

iii) Suppose $X_f = X$. Then f is not contained in any prime ideal of X. Since every nonunit is contained in a prime ideal of A, f must be unit.

Conversely, suppose f is a unit. Since every prime ideal in A is proper, f is not in any prime ideal of A. Hence $X_f = X$.

iv) It is clear that $X_f = X_g$ if and only if $V(f) = V(g)$. Further, $V(f) = V(g)$ if and only if the set of prime ideals containing f equals the set of prime ideals containing g, which of course is equivalent to $\text{rad}(f) = \text{rad}(g)$.

v) Let $\mathscr{U}$ be an open cover of $X_f = \text{Spec}(A_f)$. Without loss of generality, we may assume $\mathscr{U}$ is a cover by basic open sets X_{f_α}. Since $X_f \subseteq \bigcup_\alpha X_{f_\alpha}$, we must have that the set $\{f_\alpha\}$ generates A_f. So, there exists $g_{\alpha_1}, \ldots, g_{\alpha_n}$ so that
$$\sum_{k=1}^{n} f_{\alpha_k} g_{\alpha_k} = 1.$$

We claim that $X_{f_{\alpha_1}}, \dots, X_{f_{\alpha_n}}$ is a subcover of $\mathscr{U}$.

Assume $\mathfrak{p}$ is a prime ideal in A_f which lies in no $X_{f_{\alpha_k}}$. Then $f_{\alpha_k} \in \mathfrak{p}$ for all $k = 1, \dots, n$, and hence

$$\sum_{k=1}^{n} f_{\alpha_k} g_{\alpha_k} = 1 \in \mathfrak{p},$$

which is absurd. Hence, the collection $X_{f_{\alpha_1}}, \dots, X_{f_{\alpha_n}}$ is a subcover of $\mathscr{U}$, as desired. Since $\mathscr{U}$ is an arbitrary open cover of X_f, X_f is quasi-compact.

vi) This is proved in v).

vii) ($\Rightarrow$) Let U be a quasi-compact open subset of X. Since the basic open sets X_f form a basis for the Zariski topology, we can find an open cover $\{X_{f_\alpha}\}$ of U with $X_{f_\alpha} \subseteq U$ for all f_α. Since U is quasi-compact, we can find a finite subcover $\{X_{f_{\alpha_1}}, \dots, X_{f_{\alpha_n}}\}$. But then

$$U = \bigcup_{i=1}^{n} X_{f_{\alpha_i}}.$$

($\Leftarrow$) Let O be an open subset which is a finite union of basic open sets X_f. Say $O = \cup_{i=1}^{n} X_{f_i}$. Let $\mathscr{U}$ be a cover of O. Since the basic open sets X_f form a basis for the Zariski topology, for each point in each open set in $\mathscr{U}$, there exists a basic open set containing that point and contained in the open set. We look at that cover $\{X_{g_\alpha}\}$ of X_f by basic open sets. Since $\{X_{g_\alpha}\}$ is an open cover for each X_{f_i} and X_{f_i} is quasi-compact, we can find a finite subcover $\{X_{g_{\alpha_1}}, \dots, X_{g_{\alpha_m}}\}$. Since there are only finitely many X_{f_i}'s, doing this for each X_{f_i} provides a finite subcover of $\{X_{g_\alpha}\}$ for O itself. For each basic open set in this finite subcover, choose an open set $U_i \in \mathscr{U}$ containing it. Then the cover $\{U_i\}$ is a finite subcover of $\mathscr{U}$. Since $\mathscr{U}$ is an arbitrary cover of O, O is quasi-compact.

□

Exercise 18. For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \operatorname{Spec}(A)$. When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$ (logically, of course, it is the same thing). Show that

i) the set $\{x\}$ is closed (we say that x is a "closed point") in $\operatorname{Spec}(A)$ if and only if $\mathfrak{p}_x$ is maximal;

ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$;

iii) $y \in \overline{\{x\}}$ if and only if $\mathfrak{p}_x \subseteq \mathfrak{p}_y$;

iv) X is a T_0-space (this means that if x, y are distinct points of X, then either there is a neighborhood of x which does not contain y, or else there is a neighborhood of y which does not contain x).

Solution. *Proof.* Let $X = \mathrm{Spec}(A)$ for some ring A. For $x \in X$, let $\mathfrak{p}_x$ denote the corresponding prime ideal of A. Let $x \in X$.

ii) By definition, the closure of $\{x\}$ is the intersection of all closed sets containing x. Any closed set containing x is of the form $V(\mathfrak{a})$ where $\mathfrak{a} \subseteq \mathfrak{p}_x$. Hence, the closure of $\{x\}$ equals

$$\overline{\{x\}} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}_x} V(\mathfrak{a}) = V\left(\bigcup_{\mathfrak{a} \subseteq \mathfrak{p}_x} \mathfrak{a}\right) = V(\mathfrak{p}_x).$$

i) By part (ii), the set $\{x\}$ is closed if and only if $V(\mathfrak{p}_x) = \{\mathfrak{p}_x\}$, that is, if and only if the only prime ideal to contain $\mathfrak{p}_x$ is $\mathfrak{p}_x$. This occurs if and only if the ideal $\mathfrak{p}_x$ is maximal.

iii) We have that $y \in \overline{\{x\}}$ if and only if $\mathfrak{p}_y \in \overline{\{x\}} = V(\mathfrak{p}_x)$ if and only if $\mathfrak{p}_y \supseteq \mathfrak{p}_x$.

iv) Let x, $y \in X$ be distinct points. Since x and y are distinct points, we must have either $\mathfrak{p}_x \nsubseteq \mathfrak{p}_y$ or $\mathfrak{p}_y \nsubseteq \mathfrak{p}_x$. If $\mathfrak{p}_x \nsubseteq \mathfrak{p}_y$, then $\mathfrak{p}_y \notin V(\mathfrak{p}_x) = \overline{\{x\}}$, so $X \setminus \overline{\{x\}}$ is an open set containing y but not x. Similarly, if $\mathfrak{p}_y \nsubseteq \mathfrak{p}_x$, then $\mathfrak{p}_x \notin V(\mathfrak{p}_y) = \overline{\{y\}}$, so $X \setminus \overline{\{y\}}$ is an open set containing x but not y. Since x, y are arbitrary distinct points in X, X is a T_0-space.

□

Exercise 19. A topological space X is said to be *irreducible* if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that $\mathrm{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Solution. *Proof.* We note that X being irreducible is equivalent to that fact that X cannot be written as the union of two proper, nonempty, closed subsets.

Let A be a ring.

($\Leftarrow$) Suppose the nilradical $\mathfrak{N}_A$ of A is a prime ideal. Then $\mathfrak{N}_A \in \mathrm{Spec}(A)$ and $\overline{\mathfrak{N}_A} = V(\mathfrak{N}_A) = \mathrm{Spec}(A)$. Hence the closure of $\mathfrak{N}_A$ is $\mathrm{Spec}(A)$.

Let U, V be nonempty open sets in X. Assume $U \cap V = \emptyset$. Then

$$X = X \setminus \emptyset = X \setminus U \cap V = X \setminus U \cup X \setminus V.$$

Since $\mathfrak{N}_A \in X$, $\mathfrak{N}_A$ must be contained in one of these two closed sets. Since $\mathfrak{N}_A$ is a point whose closure is X, this forces U or V to be empty. This contradiction shows that $U \cap V \neq \emptyset$, so X is irreducible.

($\Rightarrow$) Suppose the nilradical $\mathfrak{N}_A$ is not prime. Then there exist $f, g \in A$ so that $f, g \notin \mathfrak{N}_A$, but $fg \in \mathfrak{N}_A$. Let $U = X_f$ and $V = X_g$. Since $f \notin \mathfrak{N}_A$, there exists some prime ideal that does not contain f. Hence U is nonempty. For the same reason, V is nonempty. Further

$$U \cap V = X_f \cap X_g = X_{fg} = \emptyset,$$

since fg is nilpotent and hence contained in every prime ideal. So, we see X is reducible. □

Exercise 20. Let X be a topological space.

i) If Y is an irreducible (Exercise 19) subspace of X, then the closure $\overline{Y}$ of Y in X is irreducible.

ii) Every irreducible subspace of X is contained in a maximal irreducible subspace.

iii) The maximal irreducible subspaces of X are closed and cover X. They are called the *irreducible components* of X. What are the irreducible components of a Hausdorff space?

iv) If A is a ring and $X = \operatorname{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A.

Solution. *Proof.* Let X be a topological space.

i) Let $Y \subseteq X$ be irreducible and consider the closure $\overline{Y}$ of Y. Suppose U and V are nonempty, open sets in $\overline{Y}$ and assume $U \cap V = \emptyset$. Then $\emptyset = (U \cap V) \cap Y = (U \cap Y) \cap (V \cap Y)$. Since Y is irreducible, one of these sets must be empty.

   Suppose $U \cap Y$ is empty. Then $Y \subseteq (\overline{Y} \setminus U)$, which is a proper, closed set in $\overline{Y}$. This contradicts the definition of $\overline{Y}$, so we see that $U \cap V \neq \emptyset$. Similarly, if $V \cap Y$ is empty, we have $U \cap V \neq \emptyset$. Hence, any two nonempty open sets in $\overline{Y}$ intersect. It follows that $\overline{Y}$ is irreducible.

ii) Let Y be an irreducible subspace of X. Let $\mathscr{F}$ be the family of irreducible subspaces which contain Y. Note that $\mathscr{F}$ is nonempty, since $Y \in \mathscr{F}$.

Let $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ be a chain in $\mathscr{F}$. Let $S = \bigcup_{i=1}^{\infty} X_i$. We claim that S is irreducible and contains Y. Clearly S contains Y. Suppose U and V be nonempty open subsets of S. Since U is a a nonempty subset of S, $U \cap X_i \neq \emptyset$ for some i. Similarly, since V is a a nonempty subset of S, $V \cap X_j \neq \emptyset$ for some j. Let $k = \max\{i, j\}$. Then $U \cap X_k$ and $V \cap X_k$ are nonempty open sets in X_k. Since X_k is irreducible, these two open sets must have nonempty intersection. But then $U \cap V$ is nonempty. This shows S is irreducible.

By Zorn's lemma, $\mathscr{F}$ contains maximal elements. So, there exists a maximal irreducible set containing Y.

iii) Let A be a ring and $X = \operatorname{Spec}(A)$. Let $x \in X$. Since $\{x\}$ is irreducible, by part (ii), it must be contained in a maximal irreducible subspace of X. It follows that X is covered by maximal irreducible subspaces. Let Y be a maximal irreducible subspace of X. By part (i), $\overline{Y}$ is also irreducible. By the maximality of Y, we must have $Y = \overline{Y}$, so Y is closed.

If X is a Hausdorff topological space, the irreducible components are sets consisting of single points. Indeed, given two (distinct) points x_1 and $x_2 \in X$, since X is Hausdorff, we can find disjoint open sets U, V so that $x_1 \in U$ and $x_2 \in V$. It follows that any Hausdorff space with at least two points is necessarily reducible. Hence, the maximal irreducible components of a Hausdorff space are sets consisting of singleton points.

iv) Let A be a ring and $X = \operatorname{Spec}(A)$. Let $Y \subseteq X$ be an irreducible component. Since Y is closed by (iii), we must have $Y = V(\mathfrak{p})$ for some ideal $\mathfrak{p} \in A$. Since $Y \subseteq X$ is irreducible, by Exercise 19, $\mathfrak{p}$ is prime. Suppose $\mathfrak{q} \subsetneq \mathfrak{p}$ is likewise a prime ideal. Then $Y = V(\mathfrak{p}) \subsetneq V(\mathfrak{q})$, and since $\mathfrak{q}$ is prime, $V(\mathfrak{q})$ is irreducible. This contradicts the fact that Y is an irreducible component. Thus, irreducible components correspond to minimal prime ideals in A.

$\square$

Exercise 21. Let $\phi : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A, i.e., a point of X. Hence ϕ induces a mapping $\phi^* : Y \to X$. Show that

i) If $f \in A$ then ${\phi^*}^{-1}(X_f) = Y_{\phi(f)}$, and hence that ϕ^* is continuous.

ii) If $\mathfrak{a}$ is an ideal of A, then ${\phi^*}^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

iii) If $\mathfrak{b}$ is an ideal of B, then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\operatorname{Ker}(\phi))$ of X. (In particular, $\operatorname{Spec}(A)$ and $\operatorname{Spec}(A/\mathfrak{N})$ (where $\mathfrak{N}$ is the nilradical of A) are naturally homeomorphic.)

v) If ϕ is injective, then $\phi^*(Y)$ is dense in X. More precisely, $\phi^*(Y)$ is dense in $X \iff \mathrm{Ker}\,(\phi) \subseteq \mathfrak{N}$.

vi) Let $\psi : B \to C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

vii) Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be the field of fractions of A. Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \to B$ by $\phi(x) = (\overline{x}, x)$, where $\overline{x}$ is the image of x in $A/\mathfrak{p}$. Show that ϕ^* is bijective but not a homeomorphism.

Solution. *Proof.* Let $\phi : A \to B$ be a ring homomorphism. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A, i.e., a point of X. Hence ϕ induces a mapping $\phi^* : Y \to X$.

i) Let $f \in A$. Let $y \in {\phi^*}^{-1}(X_f)$. Then $\phi^*(y) = \phi^{-1}(\mathfrak{p}_y) \in X_f$, so $\phi(f) \notin \mathfrak{p}_y$. Hence, $y \in Y_{\phi(f)}$. Since $y \in {\phi^*}^{-1}(X_f)$ is arbitrary, ${\phi^*}^{-1}(X_f) \subseteq Y_{\phi(f)}$.

   On the other hand, suppose $y \in Y_{\phi(f)}$. Then $\phi(f) \notin \mathfrak{p}_y$, so $f \notin \phi^{-1}(\mathfrak{p}_y) = \phi^*(y)$. Since $f \notin \phi^*(y)$, $\phi^*(y) \in X_f$, whereby, $y \in {\phi^*}^{-1}(X_f)$. Since $y \in Y_{\phi(f)}$ is arbitrary, $Y_{\phi(f)} \subseteq {\phi^*}^{-1}(X_f)$.

   Combining these two inclusions gives ${\phi^*}^{-1}(X_f) = Y_{\phi(f)}$. Since the basic open sets form a basis for the Zariski topology, this is sufficient to show that ϕ^* is continuous.

ii) Let $\mathfrak{a}$ be an ideal of A.

   Let $y \in {\phi^*}^{-1}(V(\mathfrak{a}))$. Then $\phi^*(y) = \phi^{-1}(\mathfrak{p}_y) \in V(\mathfrak{a})$, whereby $\mathfrak{a} \subseteq \phi^{-1}(\mathfrak{p}_y)$. Hence $\phi(\mathfrak{a}) \subseteq \mathfrak{p}_y$. Since $\phi(\mathfrak{a}) \subseteq \mathfrak{p}_y$ and $\mathfrak{a}^e$ is the ideal in B generated by $\phi(\mathfrak{a})$, we must have $\mathfrak{a}^e \subseteq \mathfrak{p}_y$. Hence, $y \in V(\mathfrak{a}^e)$. Since $y \in {\phi^*}^{-1}(V(\mathfrak{a}))$ is arbitrary, ${\phi^*}^{-1}(V(\mathfrak{a})) \subseteq V(\mathfrak{a}^e)$.

   On the other hand, let $y \in V(\mathfrak{a}^e)$. Then $\mathfrak{a}^e \subseteq \mathfrak{p}_y$, whereby $\phi(\mathfrak{a}) \subseteq \mathfrak{a}^e \subseteq \mathfrak{p}_y$. Hence $\mathfrak{a} \subseteq \phi^{-1}\mathfrak{p}_y$, so $\phi^*(y) = \phi^{-1}\mathfrak{p}_y \in V(\mathfrak{a})$, and $y \in {\phi^*}^{-1}(V(\mathfrak{a}))$. Since $y \in V(\mathfrak{a}^e)$ is arbitrary, we have $V(\mathfrak{a}^e) \subseteq {\phi^*}^{-1}(V(\mathfrak{a}))$.

   Combining these two inclusions, we have ${\phi^*}^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

iii) Let $\mathfrak{b}$ be an ideal of B. Let $x \in \phi^*(V(\mathfrak{b}))$. Then $\phi(\mathfrak{p}_x) = {\phi^*}^{-1}(x) \in V(\mathfrak{b})$, whereby $\mathfrak{b} \subseteq \phi(\mathfrak{p}_x)$. So, $\mathfrak{b}^c \subseteq \mathfrak{p}_x$, whereby $x \in V(\mathfrak{b}^c)$. Since $x \in \phi^*(V(\mathfrak{b}))$ is arbitrary, $\phi^*(V(\mathfrak{b})) \subseteq V(\mathfrak{b}^c)$. Since $V(\mathfrak{b}^c)$ is closed, $\overline{\phi^*(V(\mathfrak{b}))} \subseteq V(\mathfrak{b}^c)$.

   On the other hand, let $x \in V(\mathfrak{b}^c)$. Then $\mathfrak{b}^c = \phi^{-1}(\mathfrak{b}) \subseteq \mathfrak{p}_x$, whereby $\mathfrak{b} \subseteq \phi(\mathfrak{p}_x) = {\phi^*}^{-1}(x)$. Thus, ${\phi^*}^{-1}(x) \in V(\mathfrak{b})$, whereby $x \in \phi^*(V(\mathfrak{b}))$. Since $x \in V(\mathfrak{b}^c)$ is arbitrary, $V(\mathfrak{b}^c) \subseteq \phi^*(V(\mathfrak{b}))$. Putting the two inclusions together yields

$$\overline{\phi^*(V(\mathfrak{b}))} \subseteq V(\mathfrak{b}^c) \subseteq \phi^*(V(\mathfrak{b})),$$

   which implies that

$$\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c).$$

iv) Suppose $\phi : A \to B$ is surjective. If $\mathfrak{a}$ is the kernel of ϕ, then ϕ induces an isomorphism $A/\mathfrak{a} \cong B$, so $\phi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induces a homeomorphism $\mathrm{Spec}(A/\mathfrak{a}) \cong \mathrm{Spec}(B)$. Since $\mathrm{Spec}(A/\mathfrak{a}) \cong V(\mathfrak{a}) = V(\mathrm{Ker}\,(\phi))$, this concludes the proof.

v) Suppose $\phi : A \to B$ is injective. Suppose $F = V(\mathfrak{a})$ is a closed set in $\mathrm{Spec}(A)$ so that $\phi^*(\mathrm{Spec}(B)) \subseteq F$. Since $\phi^*(\mathrm{Spec}(B)) \subseteq V(\mathfrak{a})$, we have that $\phi^{-1}(\mathfrak{b}) \supseteq \mathfrak{a}$ for all prime ideals $\mathfrak{b} \subseteq B$. So, $\phi(\mathfrak{a}) \subseteq \mathfrak{b}$ for all prime ideals $\mathfrak{b} \subseteq B$.

Let $x \in \mathfrak{a}$. Since $\phi(\mathfrak{a}) \subseteq \mathfrak{b}$ for all prime ideals $\mathfrak{b} \subseteq B$, $\phi(x)$ is nilpotent in B. Consequently, $\phi(x^n) = 0$ for some natural number $n \in \mathbb{N}$. Since ϕ is injective, we have that $x^n = 0$ in A, whereby x is nilpotent. Since $x \in \mathfrak{a}$ is arbitrary, every element of $\mathfrak{a}$ is nilpotent. Consequently, $\mathfrak{a}$ is contained in every prime ideal of A. So, $F = V(\mathfrak{a}) = \mathrm{Spec}(A)$. So, $\phi^*(\mathrm{Spec}(B))$ is dense in $\mathrm{Spec}(A)$.

More generally, suppose that $\mathrm{Ker}\,(\phi) \subseteq \mathfrak{N}$, where $\mathfrak{N}$ is the nilradical of A. Let $x \in \mathfrak{a}$. As above, $\phi(x)$ is nilpotent in B, so $\phi(x^n) = \phi(x)^n = 0$ for some $n \in \mathbb{N}$. Hence, $x^n \in \mathrm{Ker}\,(\phi) \subseteq \mathfrak{N}$, whereby x is nilpotent in A. So, every element of $\mathfrak{a}$ is nilpotent. The proof now proceeds as above.

Conversely, suppose $\phi^*(\mathrm{Spec}(B))$ is dense in $\mathrm{Spec}(A)$. This means every prime ideal $\mathfrak{p}$ in A contains $\phi^*(\mathrm{Spec}(B)) = \phi^{-1}(\mathrm{Spec}(B))$.

Let $\mathfrak{p} \subset A$ be a prime ideal in A and let $\mathfrak{q}$ be a prime ideal in B. Then $\phi^{-1}(\mathfrak{q}) \subset \mathfrak{p}$.

Let $x \in \mathrm{Ker}\,(\phi)$. Then $\phi(x) = 0 \in \mathfrak{q}$ so that $x \in \phi^{-1}(\mathfrak{q}) \subset \mathfrak{p}$. Since this is true for all prime ideals $\mathfrak{p}$ in A, $x \in \mathfrak{N}$, the nilradical of A. Since $x \in \mathrm{Ker}\,(\phi)$ is arbitrary, $\mathrm{Ker}\,(\phi) \subseteq \mathfrak{N}$.

vi) Let $\psi : B \to C$ be another ring homomorphism. Then $\psi \circ \phi : A \to C$ induces $(\psi \circ \phi)^* : \mathrm{Spec}(C) \to \mathrm{Spec}(A)$. Let $\mathfrak{p}$ be a prime ideal in C. Then

$$\begin{aligned}(\psi \circ \phi)^*(\mathfrak{p}) &= (\psi \circ \phi)^{-1}(\mathfrak{p}) \\ &= \phi^{-1}(\psi^{-1}(\mathfrak{p})) \\ &= \phi^*(\psi^*(\mathfrak{p})) \\ &= (\phi^* \circ \psi^*)(\mathfrak{p}).\end{aligned}$$

So, $(\psi \circ \phi)^* = \phi^* \circ \psi^*$, as desired.

vii) Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be the field of fractions of A. Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \to B$ by $\phi(x) = (\overline{x}, x)$, where $\overline{x}$ is the image of x in $A/\mathfrak{p}$. Consider $\phi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

We note that both $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ contain two points. $\mathrm{Spec}(A)$ consists of the two points, (0) and $\mathfrak{p}$, the first of which is open and dense and the second of which is closed. $\mathrm{Spec}(B)$ consists of the two points, $(1,0)$ and $(0,1)$, each of which is closed.

Then

$$\phi^*(1,0) = (0) \tag{1.1}$$
$$\phi^*(0,1) = \mathfrak{p}. \tag{1.2}$$

So, ϕ is a bijection between $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$. However, this map is not a homeomorphism since the two spaces are not homeomorphic, as is seen by comparing the two respective topologies. (Or, you can say that ϕ^* takes the closed point $(1,0)$ to the open, dense point (0).)

$\square$

Exercise 22. Let $A = \prod_{i=1}^n A_i$ be the direct product of rings A_i. Show that $\mathrm{Spec}(A)$ is the disjoint union of open (and closed) subspaces X_i, where X_i is canonically homeomorphic with $\mathrm{Spec}(A_i)$.

Conversely, let A be any ring. Show that the following statements are equivalent:

i) $X = \mathrm{Spec}(A)$ is disconnected.

ii) $A \cong A_1 \times A_2$ where neither of the rings A_1, A_2 is the zero ring.

iii) A contains an idempotent $\neq 0$, 1.

In particular, the spectrum of a local ring is always connected.

Solution. *Proof.* Suppose a ring A is a product of rings $A_1, \ldots, A_n$, so that $A = \prod_{i=1}^n A_i$.

For each k, $1 \leq k \leq n$, let $\phi_k : A \to A_k$ be the canonical projection map and let $\mathfrak{b}_k = \mathrm{Ker}\,(\phi_k)$. Note that $\mathfrak{b}_k = \{(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n) : a_i \in A_i\}$

The canonical ring homomorphism π_k is a surjective ring homomorphism, so by Exercise 21(iv), $\pi_k^* : \mathrm{Spec}(A_k) \to \mathrm{Spec}(A)$ is a homeomorphism from $\mathrm{Spec}(A_k)$ onto its image, which is the closed set $X_k := V(\mathfrak{b}_k)$.

I claim that $\mathrm{Spec}(A) = \amalg_{k=1}^n X_k$.

Since $X_k \subseteq \mathrm{Spec}(A)$ for all k, we have $\bigcup_{k=1}^n X_k \subseteq \mathrm{Spec}(A)$.

Let $\mathfrak{p} \in X_k \cap X_\ell$ for $k \neq \ell$. Then $\mathfrak{p} \supseteq \mathfrak{b}_k$ and $\mathfrak{p} \supseteq \mathfrak{b}_\ell$, so $\mathfrak{p} \supseteq (\mathfrak{b}_k \cup \mathfrak{b}_\ell)$. But the set $\mathfrak{b}_k \cup \mathfrak{b}_\ell$ generates the entire ring A, which implies that $\mathfrak{p} \supseteq A$, which is not possible. Hence, $X_k \cap X_\ell = \emptyset$. So, the subsets $X_1, \ldots, X_n$ are disjoint.

Let $\mathfrak{p}$ be a prime ideal in A. Then $\mathfrak{p} \supseteq \{0\} = \bigcap_{k=1}^n \mathfrak{b}_k$. Since $\mathfrak{p}$ is a prime ideal, we must have that $\mathfrak{p} \supseteq \mathfrak{b}_k$ for some k, so $\mathfrak{p} \in X_k$. It follows that $\mathrm{Spec}(A) = \amalg_{k=1}^n X_k$, as desired.

ii)⇒iii) Suppose $A \cong A_1 \times A_2$ where neither of the rings A_1, A_2 is the zero ring. Let e_{A_1} be the multiplicative identity in A_1. Then let e be the element in A corresponding to $(e_{A_1}, 0)$ under the isomorphism. Then $e^2 = e$ and e is not equal to 0 or 1 in A.

iii)⇒i) Suppose A contains an idempotent $e \neq 0$, 1. Let U be the prime ideals containing e and let V be the prime ideals containing $1-e$. Hence, both U and V are closed in $\operatorname{Spec}(A)$.

Suppose $\mathfrak{p} \in U \cap V$. Then $\mathfrak{p}$ contains both e and $1-e$, which implies that $\mathfrak{p}$ contains 1, which is not possible. So U and V are disjoint. On the other hand, $e(1-e) = e - e^2 = e - e = 0$, so $0 = e(1-e) \in \mathfrak{p}$ for all prime ideals $\mathfrak{p}$. Hence, every prime ideal in A contains either e or $1-e$, so every prime ideal is in U or V. Consequently, $\operatorname{Spec}(A) = U \amalg V$.

iii)⇒ii) Suppose A contains an idempotent $e \neq 0$, 1. We note that $1-e$ is also an idempotent not equal to 0 or 1 and that $e(1-e) = 0$. Define the ideals $\mathfrak{a} = (e)$ and $\mathfrak{b} = (1-e)$. Consider the rings $A_1 = A/\mathfrak{a}$ and $A_2 = A/\mathfrak{b}$. For $x \in A$, $x = xe + x(1-e)$, so $\mathfrak{a} + \mathfrak{b} = (1)$. Let $y \in \mathfrak{a} \cap \mathfrak{b}$. Then

$$y = ae = b(1-e), \tag{1.3}$$

for some a, $b \in A$. Multiplying Equation (1.3) by e, we get $ae^2 = ae = y = 0$, so $\mathfrak{a} \cap \mathfrak{b} = (0)$. It follows that $A \cong A_1 \times A_2$.

i)⇒iii) Suppose $\operatorname{Spec}(A)$ is disconnected. Then there exist closed (and open) sets $V(\mathfrak{a})$ and $V(\mathfrak{b})$ so that $X = V(\mathfrak{a}) \amalg V(\mathfrak{b})$. So, every prime ideal in A contains either $\mathfrak{a}$ or $\mathfrak{b}$ and no prime ideal contains both.

Since no prime ideal contains both $\mathfrak{a}$ and $\mathfrak{b}$, we must have that $\mathfrak{a} + \mathfrak{b} = (1)$. So there exist $e \in \mathfrak{a}$ and $f \in \mathfrak{b}$ so that $e + f = 1$. Then $e \neq 0, 1$ and $e(1-e) = ef \in \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{N}$, the nilradical of A. Then e is idempotent in $A/\mathfrak{N}$.

□

Exercise 23. Let A be a Boolean ring (Exercise 11) and let $X = \operatorname{Spec}(A)$.

i) For each $f \in A$, the set X_f (Exercise 17) is both open and closed in X.

ii) Let $f_1, \ldots, f_n \in A$. Show that $X_{f_1} \cup \cdots \cup X_{f_n} = X_f$ for some $f \in A$.

iii) The sets X_f are the only subsets of X which are both open and closed.

iv) X is a compact Hausdorff space.

Solution. *Proof.* Let A be a Boolean ring and let $X = \operatorname{Spec}(A)$.

i) Let $f \in A$. The set X_f is always open. Since A is a Boolean ring, f is idempotent. Then the closed sets $V(f)$ and $V(1-f)$ are disjoint and have union X. This means $V(f)$ is the complement of $V(1-f)$ in X, so $X_f = V(1-f)$, so it's also closed.

ii) Let $f_1, \ldots, f_n \in A$. Then

$$\begin{aligned} X_{f_1} \cup \cdots \cup X_{f_n} &= (\operatorname{Spec}(A) \setminus V(f_1)) \cup \cdots \cup (\operatorname{Spec}(A) \setminus V(f_n)) \\ &= \operatorname{Spec}(A) \setminus \cap_i V(f_i) \\ &= \operatorname{Spec}(A) \setminus V(f_1, \ldots, f_n) \end{aligned}$$

By Exercise 11, every finitely generated ideal in A is principal, so we can find $f \in A$ so that $(f_1, \ldots, f_n) = (f)$. Then

$$\begin{aligned} X_{f_1} \cup \cdots \cup X_{f_n} &= \operatorname{Spec}(A) \setminus V(f_1, \ldots, f_n) \\ &= \operatorname{Spec}(A) \setminus V(f) \\ &= X_f. \end{aligned}$$

iii) Let $S \subseteq X$ be both open and closed. Since X is quasi-compact, S is likewise quasi-compact. By Exercise 17(vii), S is the finite union of basic open sets, $X_{f_1}, \ldots X_{f_n}$. By part (ii), $S = X_{f_1} \cup \cdots \cup X_{f_n} = X_f$ for some $f \in A$.

iv) By Exercise 17(v), X is quasi-compact, so all we have to do is show that X is Hausdorff. Let $x_{\mathfrak{p}}$ and $x_{\mathfrak{q}}$ be distinct points in $\operatorname{Spec}(A)$. Let $f \in \mathfrak{p} \setminus \mathfrak{q}$. Since f is idempotent, we have that X is the disjoint union of X_f and X_{1-f}. Since $f \in \mathfrak{p}$, $x_{\mathfrak{p}} \notin X_f$, so $x_{\mathfrak{p}} \in X_{1-f}$. On the other hand, since $f \notin \mathfrak{q}$, $x_{\mathfrak{q}} \in X_f$. Since X_f and X_{1-f} are open and disjoint, this shows X is Hausdorff. The same result holds for $f \in \mathfrak{q} \setminus \mathfrak{p}$.

$\square$

Exercise 24. Let L be a lattice, in which the sup and inf of two elements a, b are denoted by $a \vee b$ and $a \wedge b$, respectively. L is a *Boolean lattice* (or *Boolean algebra*) if

i) L has a least element and a greatest element (denoted by 0, 1, respectively).

ii) Each of $\vee$, $\wedge$ is distributive over the other.

iii) Each $a \in L$ has a unique "complement" $a' \in L$ such that $a \vee a' = 1$ and $a \wedge a' = 0$.

(For example, the set of all subsets of a set, ordered by inclusion, is a Boolean lattice.)
Let L be a Boolean lattice. Define addition and multiplication in L by the rules

$$a + b = (a \wedge b') \vee (a' \wedge b), \quad ab = a \wedge b.$$

Verify that in this way L becomes a Boolean ring, say $A(L)$.
Conversely, starting from a Boolean ring A, define an ordering on A as follows: $a \leq b$ means that $a = ab$. Show that, with respect to this ordering, A is a Boolean lattice. [The sup and inf are given by $a \vee b = a + b + ab$ and $a \wedge b = ab$, and the complement by $a' = 1 - a$.] In this way we obtain a one-to-one correspondence between (isomorphism classes of) Boolean rings and (isomorphism classes of) Boolean lattices.

Solution. Let L be a Boolean lattice. Define addition and multiplication in L by the rules

$$a + b = (a \wedge b') \vee (a' \wedge b), \quad ab = a \wedge b. \tag{1.4}$$

For $a \in L$, we have

$$a^2 = aa = a \wedge a = a,$$

so $A(L)$ with the operations defined in Equation 1.4 is a Boolean ring.

Let A be a Boolean ring. Define an ordering $\leq$ by $a \leq b$ means that $a = ab$.

For any $x \in A$, $0 = 0x$, so $0 \leq x$. So, $(A, \leq)$ has least element. Also for any $x \in A$, $x = x1$, so $x \leq 1$. So, $(A, \leq)$ has greatest element.

Define the sup by $a \vee b = a + b + ab$. Define the inf by $a \wedge b = ab$. Define the complement by $a' = 1 - a$.

Recall that we have $2x = 0$ for all $x \in A$ by Exercise 11(a). For a, b, $c \in A$, we have

$$\begin{aligned}
a \vee (b \wedge c) &= a \vee bc \\
&= a + bc + abc \\
&= a + bc + abc + 2(ac + ab + abc) \\
&= a + ac + ac + ab + bc + abc + ab + abc + abc \\
&= a^2 + ac + a^2c + ab + bc + abc + a^2b + abc + a^2bc \\
&= (a + b + ab)(a + c + ac) \\
&= (a + b + ab) \wedge (a + c + ac) \\
&= (a \vee b) \wedge (a \vee c). \\
a \wedge (b \vee c) &= a \wedge (b + c + bc) \\
&= a(b + c + bc) \\
&= ab + ac + abc \\
&= abc + ab + ac \\
&= a^2bc + ab + ac \\
&= ab \vee ac \\
&= (a \wedge b) \vee (a \wedge c).
\end{aligned}$$

$$a \vee a' = a + a' + aa' = a + (1 - a) + a(1 - a) = 1 + a - a^2 = 1 + a - a = 1.$$

$$a \wedge a' = aa' = a(1 - a) = a - a^2 = a - a = 0.$$

This shows A with these operations is a Boolean lattice.

In this way we obtain a one-to-one correspondence between (isomorphism classes of) Boolean rings and (isomorphism classes of) Boolean lattices.

Exercise 25. From the last two exercises deduce Stone's theorem, that every Boolean lattice is isomorphic to the lattice of open-and-closed subsets of some compact Hausdorff topological space.

Solution. *Proof.* Let L be a Boolean lattice and let A be the Boolean ring associated to L as in Exercise 24. Let $X = \operatorname{Spec}(A)$, which is known to be a compact Hausdorff space from Exercise 23. Let L' be the lattice of $\{X_f \mid f \in A\}$. By Exercise 23, we have

$$\begin{aligned} X_f \vee X_g &= X_f \cup X_g = X_h \text{ for some } h \in A \\ X_f \wedge X_g &= X_f \cap X_g = X_{fg} \end{aligned}$$

Since the correspondence between (isomorphism classes of) Boolean rings and (isomorphism classes of) Boolean lattice is a one-to-one correspondence, L is isomorphic to L' and this proves the result. $\square$

Exercise 26. Let A be a ring. The subspace of $\operatorname{Spec}(A)$ consisting of the *maximal* ideals of A, with the induced topology, is called the *maximal spectrum* of A and is denoted by $\operatorname{Max}(A)$. For arbitrary commutative rings it does not have the nice functorial properties of $\operatorname{Spec}(A)$, because the inverse image of a maximal ideal under a ring homomorphism need not be maximal.

Let X be a compact Hausdorff space and let $C(X)$ denote the ring of all real-valued continuous functions on X. For each $x \in X$, let $\mathfrak{m}_x$ be the set of all $f \in C(X)$ such that $f(x) = 0$. The ideal $\mathfrak{m}_x$ is maximal, because it is the kernel of the homomorphism $C(X) \to \mathbb{R}$ which takes f to $f(x)$. If $\tilde{X}$ denotes $\operatorname{Max}(C(X))$, we have therefore defined a mapping $\mu : X \to \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

We shall show that μ is a homeomorphism of X onto $\tilde{X}$.

i) Let $\mathfrak{m}$ be any maximal ideal of $C(X)$, and let $V = V(\mathfrak{m})$ be the set of common zeros of the functions in $\mathfrak{m}$: that is,

$$V = \{x \in X \mid f(x) = 0 \text{ for all } f \in \mathfrak{m}\}.$$

Suppose that V is empty. Then for each $x \in X$ there exists $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$. Since f_x is continuous, there is an open neighborhood U_x of x in X on which f_x does not vanish. By compactness a finite number of the neighborhoods, say $U_{x_1}, \ldots, U_{x_n}$, cover X. Let

$$f = f_{x_1}^2 + \cdots + f_{x_n}^2.$$

Then f does not vanish at any point of X, hence is a unit in $C(X)$. But this contradicts $f \in \mathfrak{m}$, hence V is not empty. Let x be a point of V. Then $\mathfrak{m} \subset \mathfrak{m}_x$, hence $\mathfrak{m} = \mathfrak{m}_x$ because $\mathfrak{m}$ is maximal. Hence μ is surjective.

ii) By Urysohn's lemma (this is the only non-trivial fact required in the argument) the continuous functions separate the points of X. Hence $x \neq y \Rightarrow \mathfrak{m}_x \neq \mathfrak{m}_y$, and therefore μ is injective.

iii) Let $f \in C(X)$; let
$$U_f = \{x \in X \mid f(x) \neq 0\}$$
and let
$$\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} \mid f \notin \mathfrak{m}\}.$$
Show that $\mu(U_f) = \tilde{U}_f$. The open sets U_f (resp. $\tilde{U}_f$) form a basis of the topology of X (resp. $\tilde{X}$) and therefore μ is a homeomorphism.

Thus X can be reconstructed from the ring of functions $C(X)$.

Solution. Let A be a ring. The subspace of $\operatorname{Spec}(A)$ consisting of the *maximal* ideals of A, with the induced topology, is called the *maximal spectrum* of A and is denoted by $\operatorname{Max}(A)$. For arbitrary commutative rings it does not have the nice functorial properties of $\operatorname{Spec}(A)$, because the inverse image of a maximal ideal under a ring homomorphism need not be maximal.

Let X be a compact Hausdorff space and let $C(X)$ denote the ring of all real-valued continuous functions on X. For each $x \in X$, let $\mathfrak{m}_x$ be the set of all $f \in C(X)$ such that $f(x) = 0$. The ideal $\mathfrak{m}_x$ is maximal, because it is the kernel of the homomorphism $C(X) \to \mathbb{R}$ which takes f to $f(x)$. If $\tilde{X}$ denotes $\operatorname{Max}(C(X))$, we have therefore defined a mapping $\mu : X \to \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

i) *Proof.* Let $\mathfrak{m}$ be any maximal ideal of $C(X)$, and let $V = V(\mathfrak{m})$ be the set of common zeros of the functions in $\mathfrak{m}$: that is,
$$V = \{x \in X \mid f(x) = 0 \text{ for all } f \in \mathfrak{m}\}.$$
Suppose that V is empty. Then for each $x \in X$ there exists $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$. Since f_x is continuous, there is an open neighborhood U_x of x in X on which f_x does not vanish. By compactness a finite number of the neighborhoods, say $U_{x_1}, \ldots, U_{x_n}$, cover X. Let
$$f = f_{x_1}^2 + \cdots + f_{x_n}^2.$$
Then f does not vanish at any point of X, hence is a unit in $C(X)$. But this contradicts $f \in \mathfrak{m}$, hence V is not empty. Let x be a point of V. Then $\mathfrak{m} \subset \mathfrak{m}_x$, hence $\mathfrak{m} = \mathfrak{m}_x$ because $\mathfrak{m}$ is maximal. Hence μ is surjective. ☐

ii) *Proof.* By Urysohn's lemma the continuous functions separate the points of X. Hence $x \neq y \Rightarrow \mathfrak{m}_x \neq \mathfrak{m}_y$, and therefore μ is injective. ☐

iii) *Proof.* Let $f \in C(X)$; let
$$U_f = \{x \in X \mid f(x) \neq 0\}$$

and let

$$\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} \mid f \notin \mathfrak{m}\}.$$

By definition, $\mu(x) = \mathfrak{m}_x = \{f \in C(X) \mid f(x) = 0\}$, but for $x \in U_f$, we have $f(x) \neq 0$. We conclude that $f \notin \mathfrak{m}_x$, whereby $\mathfrak{m}_x \in \tilde{U}_f$. Since $x \in U_f$ is arbitrary, $\mu(U_f) \subseteq \tilde{U}_f$. Since every $\mathfrak{m} \in \tilde{X}$ is of the form $\mathfrak{m}_x$ for some $x \in X$, we see that $\mu(U_f) = \tilde{U}_f$. This shows that μ is a homeomorphism. □

Exercise 27. Let k be an algebraically closed field and let

$$f_\alpha(t_1, \ldots, t_n) = 0$$

be a set of polynomial equations in n variables with coefficients in k. The set X of all points $x = (x_1, \ldots, x_n) \in k^n$ which satisfy these equations is an *affine algebraic variety.* Consider the set of all polynomials $g \in k[t_1, \ldots, t_n]$ with the property that $g(x) = 0$ for all $x \in X$. This set is an ideal $I(X)$ in the polynomial ring, and is called the *ideal of the variety* X. The quotient ring

$$P(X) = k[t_1, \ldots, t_n]/I(X)$$

is the ring of polynomial functions on X, because two polynomials g, h define the same polynomial function on X if and only if $g - h$ vanishes at every point of X, that is, if and only if $g - h \in I(X)$.

Let ξ_i be the image of t_i in $P(X)$. The ξ_i $(1 \leq i \leq n)$ are the *coordinate functions* on X: if $x \in X$, then ξ_i is the ith coordinate of x. $P(X)$ is generated as a k-algebra by the coordinate functions, and is called the *coordinate ring* (or algebra) of X.

As in the previous exercise, for each $x \in X$ let $\mathfrak{m}_x$ be the ideal of all $f \in P(X)$ such that $f(x) = 0$; it is a maximal ideal of $P(X)$. Hence, if $\tilde{X} = \operatorname{Max}(P(X))$, we have defined a mapping $\mu : X \to \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

It is easy to show that μ is injective: if $x \neq y$, we must have $x_i \neq y_i$ for some i $(1 \leq i \leq n)$, and hence $\xi_i - x_i$ is in $\mathfrak{m}_x$, but not in $\mathfrak{m}_y$, so that $\mathfrak{m}_x \neq \mathfrak{m}_y$. What is less obvious (but still true) is that μ is surjective. This is one form of Hilbert's Nullstellensatz.

Solution. *Proof.* Let k be an algebraically closed field and let

$$f_\alpha(t_1, \ldots, t_n) = 0$$

be a set of polynomial equations in n variables with coefficients in k.

Consider the set of all polynomials $g \in k[t_1, \ldots, t_n]$ with the property that $g(x) = 0$ for all $x \in X$. This set is an ideal $I(X)$ in the polynomial ring. Let $P(X) = k[t_1, \ldots, t_n]/I(X)$ be the coordinate ring of X. Let ξ_i be the image of t_i in $P(X)$. For each $x \in X$ let $\mathfrak{m}_x$ be the ideal of all $f \in P(X)$ such that

$f(x) = 0$; it is a maximal ideal of $P(X)$. Hence, if $\tilde{X} = \mathrm{Max}(P(X))$, we have defined a mapping $\mu : X \to \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

It is easy to show that μ is injective: if $x \neq y$, we must have $x_i \neq y_i$ for some i $(1 \leq i \leq n)$, and hence $\xi_i - x_i$ is in $\mathfrak{m}_x$, but not in $\mathfrak{m}_y$, so that $\mathfrak{m}_x \neq \mathfrak{m}_y$. What is less obvious (but still true) is that μ is surjective. This is one form of Hilbert's Nullstellensatz.

Let $\mathfrak{m}$ be a maximal ideal in $P(X)$. Suppose the affine variety defined by $\mathfrak{m}$ is empty. Then for each $x \in X$, there is a regular function $f_x \in \mathfrak{m}$ so that $f_x(x) \neq 0$. Since each f_x is continuous, for each $x \in X$, there is an open set U_x so that $f_x(y) \neq 0$ for all $y \in U_x$. Now, the set $\{U_x \,|\, x \in X\}$ is an open cover of X, which is quasi-compact in the Zariski topology, so we can find a finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ of X.

Let $f = f_1^2 + \cdots + f_n^2$. Then for any $y \in X$, we have

$$f(y) = f_1^2(y) + \cdots + f_n^2(y) \neq 0.$$

So, f is a unit in $P(X)$. However, $f \in \mathfrak{m}$, which is a contradiction.

Thus, the algebraic variety defined by $\mathfrak{m}$ is not empty. Let x be in the variety defined by $\mathfrak{m}$. If $\mathfrak{m}_x$ is the maximal ideal of x, then we have $\mathfrak{m} \subset \mathfrak{m}_x$. Since $\mathfrak{m}$ is a maximal ideal by hypothesis, $\mathfrak{m} = \mathfrak{m}_x$. This shows μ is surjective. $\square$

Exercise 28. Let $f_1, \ldots, f_m$ be elements of $k[t_1, \ldots, t_n]$. They determine a *polynomial mapping* $\phi : k^n \to k^m$: if $x \in k^n$, the coordinates of $\phi(x)$ are $f_1(x), \ldots, f_m(x)$.

Let X, Y be affine algebraic varieties in k^n, k^m respectively. A mapping $\phi : X \to Y$ is said to be *regular* if ϕ is the restriction to X of a polynomial mapping from k^n to k^m.

If η is a polynomial function on Y, then $\eta \circ \phi$ is a polynomial function on X. Hence ϕ induces a k-algebra homomorphism $P(Y) \to P(X)$, namely $\eta \mapsto \eta \circ \phi$. Show that in this way we obtain a one-to-one correspondence between regular mappings $X \to Y$ and the k-algebra homomorphisms $P(Y) \to P(X)$.

Solution. *Proof.* Let $\phi : X \to Y$ be a regular map from X to Y. Define $\phi^* : P(Y) \to P(X)$ by $\phi^*(f) = f \circ \phi$. Then

$$\begin{aligned}
\phi^*(f+g) &= (f+g) \circ \phi = f \circ \phi + g \circ \phi = \phi^*(f) + \phi * (g) \\
\phi^*(fg) &= (fg) \circ \phi = (f \circ \phi) \cdot (g \circ \phi) = \phi^*(f) \cdot \phi^*(g).
\end{aligned}$$

Hence, ϕ^* is a ring homomorphism.

Define $\Psi : \mathrm{Mor}(X, Y) \to \mathrm{Hom}(P(Y), P(X))$ by $\Psi(\phi) = \phi^*$.

Let ξ_i, $1 \leq i \leq n$, be the coordinate functions on Y. That is, ξ_i is the image of t_i in $P[Y]$.

Suppose $\Psi(\phi) = \Psi(\theta)$. Then $\phi^* = \theta^*$, so that

$$(\phi^*(\xi_i))(x) = (\theta^*(\xi_i))(x)$$
$$\xi_i(\phi(x)) = \xi_i(\theta(x)).$$

for all $x \in X$ and all i, $1 \leq i \leq n$. This means that the ith coordinate of $\phi(x)$ and the ith coordinate of $\theta(x)$ are equal for all $x \in X$ and all i, $1 \leq i \leq n$. Hence $\phi(x) = \theta(x)$ for all $x \in X$, whereby $\phi = \theta$. This proves Ψ is injective.

Let $\mu \in \mathrm{Hom}(P(Y), P(X))$. Let $f_i = \mu(\xi_i) \in P(X)$. Define $f : X \to Y$ by $f(x) = (f_1(x), \ldots, f_n(x))$. Since $f_i \in P(X)$ for all i, $f \in \mathrm{Mor}(X, Y)$. Then $(\Psi(f))(\xi_i) = f^*(\xi_i)$. So, for all $x \in X$, we have

$$(\Psi(f))(\xi_i)(x) = f^*(\xi_i)(x) = \xi_i(f(x)) = f_i(x) = \mu(\xi_i)(x).$$

So, we see that $\Psi(f) = \mu$. This shows Ψ is surjective. $\square$

Chapter 2

Modules

Exercises

Exercise 1. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

Proof. Let m and n be coprime natural numbers. Then there exist integers a and b so that $am + bn = 1$. For $\overline{x} \otimes \overline{y} \in (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$, we have

$$\begin{aligned}\overline{x} \otimes \overline{y} &= \overline{x(am+bn)} \otimes \overline{y} = \overline{x(bn)} \otimes \overline{y} \\ &= \overline{xb} \otimes \overline{ny} = \overline{xb} \otimes \overline{0} = 0.\end{aligned}$$

So, $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$. $\square$

Exercise 2. Let A be a ring, $\mathfrak{a}$ an ideal, M an A-module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. [Tensor the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ with M.]

Proof. Let A be a ring and let $\mathfrak{a}$ be an ideal in A. Consider the exact sequence of A-modules

$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0.$$

Tensoring this exact sequence with M yields

$$\mathfrak{a} \otimes_A M \to A \otimes_A M \to (A/\mathfrak{a}) \otimes_A M \to 0,$$

since the tensor product is a right-exact functor. Note that $A \otimes_A M \cong M$ by Proposition 2.14.

We will first show that $\mathfrak{a} \otimes_A M \cong M\mathfrak{a}$. Define $\phi : \mathfrak{a} \times M \to \mathfrak{a}M$ by $\phi(x, m) = xm$. This map is bilinear, so it induces a homomorphism $\phi : \mathfrak{a} \otimes_A M \to \mathfrak{a}M$ by $x \otimes m \mapsto xm$. This homomorphism is clearly surjective. Suppose $\phi(x, m) = 0$.

Then $xm = 0$, whereby $x \otimes m = 1 \otimes xm = 1 \otimes 0 = 0$. Thus, ϕ is injective and hence an isomorphism.

So, we have an exact sequence

$$\mathfrak{a}M \to M \to (A/\mathfrak{a}) \otimes_A M \to 0.$$

Now, we need to show that the first map is injective. But this is clear since the map is just inclusion. Thus, we see from the first isomorphism theorem that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$.

$\square$

Exercise 3. Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$. [Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2. By Nakayama's Lemma, $M_k = 0 \Rightarrow M = 0$. But $M \otimes_A N = 0 \Rightarrow (M \otimes_A N)_k = 0 \Rightarrow M_k \otimes_k N_k = 0 \Rightarrow M_k = 0$ or $N_k = 0$, since M_k, N_k are vector spaces over a field.]

Proof. Let A be a local ring, and let M and N finitely generated A-modules. Let $\mathfrak{m}$ be the unique maximal ideal of A and let $k = A/\mathfrak{m}$ be the residue field of $\mathfrak{m}$.

We note that $M_k = k \otimes_A M = (A/\mathfrak{m}) \otimes_A M \cong M/\mathfrak{m}M$, by Exercise 2. By Nakayama's lemma, if $M_k \cong M/\mathfrak{m}M = 0$, then $M = 0$. Now, if $M \otimes_A N = 0$, then $(M \otimes_A N)_k = 0$, whereby $M_k \otimes_k N_k = 0$. Since M_k and N_k are vector spaces over the field k, this implies that M_k or N_k is zero. It now follows by Nakayama's lemma that $M = 0$ or $N = 0$. $\square$

Exercise 4. Let M_i $(i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat if and only if each M_i is flat.

Proof. Let M_i $(i \in I)$ be any family of A-modules, and let M be their direct sum.

I claim that for any A-module N, $M \otimes N \cong \oplus(M_i \otimes N)$. Define $\phi : M \times N \to \oplus(M_i \otimes N)$ by $\phi((x_i), n) = (x_i \otimes n)$. Then ϕ is A-bilinear and so induces a homomorphism $\Phi : M \otimes N \to \oplus(M_i \otimes N)$ for which $\Phi((x_i) \otimes n) = (x_i \otimes n)$.

Let $j_i : M_i \to M$ be the natural injection. We can then define a map $M_i \times N \to M \times N \to M \otimes N$ by $(x_i, n) \mapsto j_i(x_i) \otimes n$. This map is bilinear, so we get a map $M_i \otimes N \to M \otimes N$ defined by $x_i \otimes n \mapsto j_i(x_i) \otimes n$. Together, these define a map $\Psi : \oplus(M_i \otimes N) \to M \otimes N$ given by $\Psi((x_i \otimes n_i)) = \sum j_i(x_i) \otimes n_i$ is a homomorphism.

The maps Φ and Ψ are inverses of one another, so they are isomorphisms and $M \otimes N \cong \oplus(M_i \otimes N)$.

Suppose now that $f : N' \to N$ is injective and consider the mapping $1 \otimes f : M \otimes N' \to M \otimes N$. As above, $M \otimes N'$ is isomorphic to $\oplus(M_i \otimes N')$ under Ψ', and $\oplus(M_i \otimes N)$ is isomorphic to $M \otimes N$ under Φ.

Therefore $1 \otimes f$ is injective if and only if the induced map $g = \Phi \circ (1_M \otimes f) \otimes \Psi'$ from $\oplus(M_i \otimes N')$ to $\oplus(M_i \otimes N)$ is injective.

$$\begin{tikzcd} M \otimes N' \arrow[r, "1_M \otimes f"] & M \otimes N \arrow[d, "\Phi"] \\ \oplus_i(M_i \otimes N') \arrow[u, "\Psi'"] \arrow[r, "g"] & \oplus_i(M_i \otimes N) \end{tikzcd}$$

Notice that $g((x_i \otimes n_i)) = (x_i \otimes f(n_i))$. So, $g = (1_{M_i} \otimes f)$. Hence g is injective if and only if $1_{M_i} \otimes f : M_i \otimes N \to M_i \otimes N'$ is injective for all i. That is, M is flat if and only if M_i is flat for all i. $\square$

Exercise 5. Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra.

Proof. This follows immediately from the preceding problem. The ring of polynomials $A[x]$ is isomorphic as an A-module to $\bigoplus_{i \in \mathbb{N} \cup \{0\}} A$. Since A is certainly a flat A-module, by the preceding problem, $\bigoplus_{i \in \mathbb{N} \cup \{0\}} A \cong A[x]$ is a flat A-module. $\square$

Exercise 6. For any A-module, let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form

$$m_0 + m_1 x + \cdots + m_r x^r \quad (m_i \in M).$$

Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x]$ is an $A[x]$-module.

Show that $M[x] \cong A[x] \otimes_A M$.

Proof. If we prove the second statement, the first statement follows since $A[x] \otimes_A M$ is clearly and $A[x]$-module.

Define a map $\phi : A[x] \times_A M \to M[x]$ by

$$((a_n x^n + \cdots + a_0), m) \mapsto (a_n m) x^n + \cdots + (a_0 m).$$

This map is bilinear, so it factors through the tensor product and therefore induces a homomorphism (which we also call ϕ) $\phi : A[x] \otimes M \to M[x]$. This homomorphism is clearly surjective since it is linear and surjective on monomials. Suppose $\phi(f \otimes m) = 0$. If $f(x) = a_n x^n + \cdots + a_0$, then $\phi(f \otimes m) = (a_n m)x^n + \cdots + (a_0 m) = 0$, so $a_i m = 0$ for all i, $0 \leq i \leq n$. Then

$$\begin{aligned} f \otimes m &= (a_n x^n + \cdots + a_0) \otimes m \\ &= a_n x^n \otimes m + \cdots + a_0 \otimes m \\ &= x^n \otimes a_n m + \cdots + 1 \otimes a_0 m \\ &= 0, \end{aligned}$$

since $a_i m = 0$ for all i, $0 \leq i \leq n$. So, ϕ is an isomorphism and $A[x] \otimes_A M \cong M[x]$. It follows that $M[x]$ is an $A[x]$-module. □

Exercise 7. Let $\mathfrak{p}$ be a prime ideal in A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in A, is $\mathfrak{m}[x]$ a maximal ideal in $A[x]$?

Proof. Let $\mathfrak{p}$ be a prime ideal in a ring A and consider $\mathfrak{p}[x] \subseteq A[x]$. We wish to show that $\mathfrak{p}[x]$ is a prime ideal.

The sequence of A-modules

$$0 \to \mathfrak{p} \to A \to A/\mathfrak{p} \to 0$$

is exact. By Exercise 5 of this chapter, $A[x]$ is a flat A-module. Tensoring the above exact sequence with $A[x]$, we get

$$0 \to \mathfrak{p} \otimes_A A[x] \to A \otimes_A A[x] \to (A/\mathfrak{p}) \otimes_A A[x] \to 0,$$

the sequence being exact on the left by Proposition 2.19. By Exercise 6 in this chapter, $M[x] \cong A[x] \otimes_A M$ for any A-module M. It follows that

$$0 \to \mathfrak{p}[x] \to A[x] \to (A/\mathfrak{p})[x] \to 0.$$

Hence $A[x]/\mathfrak{p}[x]$ is isomorphic to $(A/\mathfrak{p})[x]$, the ring of polynomials in one variable over an integral domain. Since this is an integral domain as well, it follows that $\mathfrak{p}[x]$ is prime in $A[x]$. □

If we substitute a maximal ideal $\mathfrak{m}$ for $\mathfrak{p}$, we see that $A[x]/\mathfrak{m}[x]$ is isomorphic to $(A/\mathfrak{m})[x]$, the ring of polynomials in one variable over a field. Since such a ring of polynomials is not necessarily a field, it is not true that $\mathfrak{m}[x]$ is a maximal ideal in $A[x]$ whenever $\mathfrak{m}$ is maximal in A.

As an example, take the ideal $\mathfrak{m} = (2)$ in $\mathbb{Z}$. This ideal is a maximal ideal and the quotient field is the field $\mathbb{Z}/2\mathbb{Z}$. Now look at $\mathfrak{a}[x] = \mathfrak{m}[x]$ in $\mathbb{Z}[x]$. Then we have ideals $\mathfrak{m}[x] \subsetneq (2, x)[x] \subsetneq \mathbb{Z}[x]$, so $\mathfrak{m}[x]$ is not a maximal ideal.

Exercise 8. (i) If M and N are flat A-modules, then so is $M \otimes_A N$.

(ii) If B is a flat A-algebra and N is a flat B-module, then N is flat as an A-module.

Proof. (i) Let M and N be flat A-modules. Let $0 \to P \to Q \to R \to 0$ be an exact sequence of A-modules. Tensoring with the flat A-module M, we get the exact sequence

$$0 \to P \otimes_A M \to Q \otimes_A M \to R \otimes_A M \to 0.$$

Tensoring with the flat A-module N, we get the exact sequence

$$0 \to (P \otimes_A M) \otimes_A N \to (Q \otimes_A M) \otimes_A N \to (R \otimes_A M) \otimes_A N \to 0.$$

Using the canonical isomorphism 2.14(ii), we get the exact sequence

$$0 \to P \otimes_A (M \otimes_A N) \to Q \otimes_A (M \otimes_A N) \to R \otimes_A (M \otimes_A N) \to 0.$$

Since the exact sequence we began with was arbitrary, $M \otimes_A N$ is a flat A-module, by Proposition 2.19.

(ii) Let B be a flat A-algebra and N be a flat B-module. Let $0 \to P \to Q \to R \to 0$ be an exact sequence of A-modules. Since B is a flat A-algebra, tensoring with B gives us the exact sequence of B-modules:

$$0 \to P \otimes_A B \to Q \otimes_A B \to R \otimes_A B \to 0.$$

Since N is a flat B-module, we can tensor this last exact sequence with N to get the exact sequence

$$0 \to (P \otimes_A B) \otimes_B N \to (Q \otimes_A B) \otimes_B N \to (R \otimes_A B) \otimes_B N \to 0.$$

Using the canonical isomorphisms in Proposition 2.15 and Proposition 2.14(iv), we get the exact sequences

$$0 \to P \otimes_A (B \otimes_B N) \to Q \otimes_A (B \otimes_B N) \to R \otimes_A (B \otimes_B N) \to 0$$
$$0 \to P \otimes_A N \to Q \otimes_A N \to R \otimes_A N \to 0.$$

Since the exact sequence we began with was arbitrary, N is a flat A-module, by Proposition 2.19. □

Exercise 9. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are finitely generated, then so is M.

Proof. Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of A-modules, where M' and M'' are finitely generated. Let $u_1, \dots, u_m$ generate M' and let $v_1, \dots, v_n$ generate M''. Let $x_i = \alpha(u_i) \in M$ for each i, $1 \leq i \leq m$. Since the map $M \to M''$ is surjective, we may choose elements $y_i \in M$ so that the image of $\beta(y_i) = v_i$ for each i, $1 \leq i \leq n$.

I claim that the set $\{x_1, \dots, x_m, y_1, \dots, y_n\}$ generates M. Let $m \in M$. Then $\beta(m) \in M''$, so we can find $a_1, \dots, a_n \in A$ so that

$$\beta(m) = a_1 v_1 + \cdots + a_n v_n.$$

Consider the element $m - (a_1 y_1 + \cdots + a_n y_n) \in M$. Then

$$\begin{aligned}\beta(m - (a_1 y_1 + \cdots + a_n y_n)) &= \beta(m) - \beta(a_1 y_1 + \cdots + a_n y_n)\\ &= \beta(m) - (a_1 \beta(y_1) + \cdots + a_n \beta(y_n))\\ &= \beta(m) - (a_1 v_1 + \cdots + a_n v_n) = 0,\end{aligned}$$

so $m - (a_1 y_1 + \cdots + a_n y_n)$ lies in the kernel of β, which equals the image of α. Thus, we can find $a'_1, \dots, a'_m \in A$ so that

$$\alpha(a'_1 u_1 + \dots a'_m u_m) = m - (a_1 y_1 + \cdots + a_n y_n).$$

Then

$$\begin{aligned}m &= \alpha(a'_1 u_1 + \dots a'_m u_m) + (a_1 y_1 + \cdots + a_n y_n)\\ &= a'_1 \alpha(u_1) + \cdots + a'_m \alpha(u_m) + (a_1 y_1 + \cdots + a_n y_n)\\ &= a'_1 x_1 + \cdots + a'_m x_m + a_1 y_1 + \cdots + a_n y_n.\end{aligned}$$

Since $m \in M$ is arbitrary, $\{x_1, \dots, x_m, y_1, \dots, y_n\}$ generates M, so M is finitely generated. □

Exercise 10. Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.

Proof. Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. Suppose the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective. Then $u(M) + \mathfrak{a}N = N$. By Nakayama's lemma, $u(M) = N$, so u is surjective. □

Exercise 11. Let A be a ring $\neq 0$. Show that $A^m \cong A^n \Rightarrow m = n$. [Let $\mathfrak{m}$ be a maximal ideal of A and let $\phi : A^m \to A^n$ be an isomorphism. Then $1 \otimes \phi : (A/\mathfrak{m}) \otimes A^m \to (A/\mathfrak{m}) \otimes A^n$ is an isomorphism between vector spaces of dimensions m and n over the field $k = A/\mathfrak{m}$. Hence $m = n$.]

Proof. Let A be a ring $\neq 0$. Let $\mathfrak{m}$ be a maximal ideal of A and let $\phi : A^m \to A^n$ be an isomorphism. Then $1 \otimes \phi : (A/\mathfrak{m}) \otimes A^m \to (A/\mathfrak{m}) \otimes A^n$ is an isomorphism between vector spaces of dimensions m and n over the field $k = A/\mathfrak{m}$. Hence $m = n$. □

Exercise 12. Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that $\operatorname{Ker}(\phi)$ is finitely generated. [Let $e_1, \ldots, e_n$ be a basis of A^n and choose $u_i \in M$ such that $\phi(u_i) = e_i$, $(1 \leq i \leq n)$. Show that M is the direct sum of $\operatorname{Ker}(\phi)$ and the submodule generated by $u_1, \ldots, u_n$.]

Proof. Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Let $e_1, \ldots, e_n$ be a basis of A^n and choose $u_i \in M$ such that $\phi(u_i) = e_i$, $(1 \leq i \leq n)$. Let $m \in M$. Since $e_1, \ldots, e_n$ is a basis of A^n, we can find $a_i \in A$, $1 \leq i \leq n$, so that $\phi(m) = a_1 e_1 + \cdots + a_n e_n$. Now, consider the element $m - (a_1 u_1 + \cdots + a_n u_n) \in M$. Then

$$\begin{aligned}
\phi(m - (a_1 u_1 + \cdots + a_n u_n)) &= \phi(m) - \phi(a_1 u_1 + \cdots + a_n u_n) \\
&= \phi(m) - (a_1 \phi(u_1) + \cdots + a_n \phi(u_n)) \\
&= \phi(m) - (a_1 e_1 + \cdots + a_n e_n) = 0,
\end{aligned}$$

so, $m - (a_1 u_1 + \cdots + a_n u_n)$ lies in the kernel of ϕ. Since $m \in M$ is arbitrary, M is spanned by $\operatorname{Ker}(\phi)$ and the set $\{u_1, \ldots, u_n\}$.

To show this is a direct sum, we show that the kernel of ϕ and the span of the set $\{u_1, \ldots, u_n\}$ meet trivially. Suppose m lies in the span of the set $\{u_1, \ldots, u_n\}$. Then $m = a_1 u_1 + \cdots + a_n u_n$ for some $a_i \in A$, $1 \leq i \leq n$. If m also lies in the kernel of ϕ, then

$$0 = \phi(m) = \phi(a_1 u_1 + \cdots + a_n u_n) = a_1 \phi(u_1) + \cdots + a_n \phi(u_n) = a_1 e_1 + \cdots + a_n e_n.$$

Since $e_1, \ldots, e_n$ are linearly independent, this implies that $a_i = 0$ for all i, $1 \leq i \leq n$, so $m = 0$. So, M is the direct sum of $\operatorname{Ker}(\phi)$ and the submodule generated by $u_1, \ldots, u_n$.

Let $m_1, \ldots, m_s$ generate M. From what we've just shown, we can write $m_i = k_i + \sum_j a^i_j u_j$, where $k_i \in \operatorname{Ker}(\phi)$.

I claim that $\operatorname{Ker}(\phi)$ is generated by $\{k_1, \ldots, k_n\}$. Let $m \in \operatorname{Ker}(\phi)$. We can write

$$\begin{aligned}
m &= \sum_i \lambda_i m_i \\
&= \sum_i \lambda_i \left(k_i + \sum_j a_j^i u_j \right) \\
&= \left(\sum_i \lambda_i k_i \right) + \left(\sum_{i,j} a_j^i u_j \right)
\end{aligned}$$

Since m and the sum $\sum_i \lambda_i k_i$ lie in the kernel of ϕ, so does their difference $m - (\sum_i \lambda_i k_i) = \sum_{i,j} a_j^i u_j$. Since the kernel of ϕ and the span of $u_1,\ldots,$ u_n meet trivially, we must have $m = \sum_i \lambda_i k_i$. Since $m \in \operatorname{Ker}(\phi)$ is arbitrary, $\operatorname{Ker}(\phi)$ is finitely generated. $\square$

Exercise 13. Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B. [Define $p : N_B \to N$ by $p(b \otimes y) = by$, and show that $N_B = \operatorname{im}(g) \oplus \operatorname{Ker}(p)$.]

Proof. Let $f : A \to B$ be a ring homomorphism and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Define the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$. Define the homomorphism $p : N_B \to N$ by $p(b \otimes y) = by$. Since $p \circ g$ is the identity map, g is injective.

Let $\sum_i b_i \otimes y_i \in N_B$. Then we can write

$$\sum_i b_i \otimes y_i = \left(\sum_i b_i \otimes y_i - g\left(\sum_i b_i y_i \right) \right) + g\left(\sum_i b_i y_i \right)$$

and

$$\begin{aligned}
p\left(\sum_i b_i \otimes y_i - g\left(\sum_i b_i y_i \right) \right) &= p\left(\sum_i b_i \otimes y_i \right) - p\left(g\left(\sum_i b_i y_i \right) \right) \\
&= \left(\sum_i p\left(b_i \otimes y_i \right) \right) - (p \circ g)\left(\sum_i b_i y_i \right) \\
&= \sum_i b_i y_i - \sum_i b_i y_i \\
&= 0.
\end{aligned}$$

So, we see that $\sum_i b_i \otimes y_i$ can be written as the sum of an element in the image of g and an element in the kernel of ϕ.

Suppose $g(n) = 1 \otimes n$ lies in the kernel of p. Then

$$0 = p(g(n)) = p(1 \otimes n) = n,$$

This shows the intersection of the image of g and the kernel of p is trivial.

So,

$$N_B = \mathrm{Im}(g) \oplus \mathrm{Ker}\,(\phi)\,.$$

Thus, $g(N)$ is a direct summand of N_B. □

Exercise 14. A partially ordered set I is said to be a *directed set* if for each pair i, j in I there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Let A be a ring, let I be a directed set and let $(M_i)_{i\in I}$ be a family of A-modules indexed by I. For each pair i, j in I such that $i \leq j$, let $\mu_{ij} : M_i \to M_j$ be an A-homomorphism, and suppose that the following axioms are satisfied:

i) μ_{ii} is the identity mapping of M_i, for all $i \in I$;

ii) $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$.

Then the modules M_i and the homomorphisms μ_{ij} are said to form a *direct system* $\mathbf{M} = (M_i, \mu_{ij})$ over the directed set I.

We shall construct an A-module M called the *direct limit* of the direct system $\mathbf{M}$. Let C be the direct sum of the M_i, and identify each module M_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \to M$ be the projection and let μ_i be the restriction of μ to M_i.

The module M, or more correctly the pair consisting of M and the family of homomorphisms $\mu_i : M_i \to M$, is called the *direct limit* of the direct system $\mathbf{M}$, and is written $\varinjlim M_i$. From the construction it is clear that $\mu_i = \mu_j \circ \mu_{ij}$. whenever $i \leq j$.

Proof. Let A be a ring, let I be a directed set, and let $\mathbf{M} = (M_i, \mu_{ij})$ be the direct system over the directed set I.

Let $C = \bigoplus_{i\in I} M_i$. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \to M$ be the projection and let μ_i be the restriction of μ to M_i. Let $\mu_{ij} : M_i \to M_j$ be the homomorphisms of the direct system. Since $\mu_{ij}(x_i) - x_i \in D$, we have that $\mu(\mu_{ij}(x_i)) = \mu(x_i)$ in M. Since $x_i \in M_i$, $\mu(x_i) = \mu_i(x_i)$. Similarly, since $\mu_{ij}(x_i) \in M_j$, $\mu(\mu_{ij}(x_i)) = \mu_j(\mu_{ij}(x_i))$. So, we have

$$\mu_i = \mu_j \circ \mu_{ij},$$

as desired. □

Exercise 15. In the situation in Exercise 14, show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$. Show also that if $\mu_i(x_i) = 0$ then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$ in M_j.

Proof. Let A be a ring, let I be a directed set, and let $\mathbf{M} = (M_i, \mu_{ij})$ be a direct system over the directed set I. Let $M = \varinjlim M_i$ and let $\mu_i : M_i \to M$ be the homomorphism defined in the previous problem.

Since M is the quotient of the direct sum $\oplus M_i$, every element of M is the image of an element $(m_i)_{m_i \in M_i}$, where all but finitely many of the m_i are zero.

Let i, j be two indices for which $m_i \neq 0$, $m_j \neq 0$. Since I is a directed set, we can find an index k so that $i \leq k$ and $j \leq k$. Then $\mu_{ik}(m_i)$ and $\mu_{jk}(m_j)$ are in M_k and

$$\begin{aligned} m_i + m_j &= (m_i - \mu_{ik}(m_i)) + (m_j - \mu_{jk}(m_j)) + \mu_{ik}(m_i) + \mu_{jk}(m_j) \\ &= \mu_{ik}(m_i) + \mu_{jk}(m_j) \text{ in } M, \end{aligned}$$

and $\mu_{ik}(m_i) + \mu_{jk}(m_j) \in M_k$. It follows by induction that we can reduce the number of indices in which the non-zero m_i's occur down to one. That is, every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$.

Now, suppose that $\mu_i(x_i) = 0$. We use the alternate equivalence relation to define the direct limit: Two elements (a_i) and (b_i) are equivalent if there exists k with $i, j \leq k$ so that $\mu_{ik}(a_i) = \mu_{jk}(b_j)$. If $\mu_i(x_i) = 0$, then (x_i) is equivalent to 0, hence there exists k with $i, j \leq k$ so that $\mu_{ik}(x_i) = \mu_{jk}(0) = 0$. $\square$

Exercise 16. Show that the direct limit is characterized (up to isomorphism) by the following property. Let N be an A-module and for each $i \in I$ let $\alpha_i : M_i \to N$ be an A-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. Then there exists a unique homomorphism $\alpha : M \to N$ such that $\alpha_i = \alpha \circ \mu_i$.

Proof. First, we note that the direct limit has this property by letting $\alpha_i = \mu_i$ and α be the identity map.

Next, let N be an A-module and for each $i \in I$ let $\alpha_i : M_i \to N$ be an A-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$.

Define $\alpha : M \to N$ as follows. Let $x \in M$. By the preceding problem, we can write $x = \mu_i(x_i)$ for some $x_i \in M_i$. Define $\alpha(x) = \alpha_i(x_i)$. Notice that this is the only way to define α so that $\alpha_i = \alpha \circ \mu_i$. So, α is unique if it is a well-defined homomorphism.

We first must show that this function is well-defined. Suppose $x = \mu_j(x_j)$ for some $x_j \in M_j$. Since I is a directed set, we can find $k \in I$ so that $i \leq k$ and $j \leq k$ so that $\mu_{ik}(x_i) = \mu_{jk}(x_j)$. Then in M, we have

$$\mu_i(x_i) = \mu_k(\mu_{ik}(x_i)) = \mu_k(\mu_{jk}(x_j)) = \mu_j(x_j).$$

Hence, $\mu_k(\mu_{ik}(x_i) - \mu_{jk}(x_j)) = 0$. By a preceding problem, we can find $m \in I$ so that $k \leq m$ and

$$\mu_{km}(\mu_{ik}(x_i) - \mu_{jk}(x_j)) = 0.$$

That is,

$$\begin{aligned}\mu_{im}(x_i) - \mu_{jm}(x_j) = 0\\ \mu_{im}(x_i) = \mu_{jm}(x_j).\end{aligned}$$

Then

$$\alpha_i(x_i) = \alpha_m(\mu_{im}(x_i)) = \alpha_m(\mu_{jm}(x_j)) = \alpha_j(x_j),$$

as desired. So α is well-defined.

Let x, $y \in M$. By the preceding problem, we can find indices i, $j \in I$ so that $x = \mu_i(x_i)$ and $y = \mu_j(x_j)$. As in proving that the map α is well-defined, we can find $k \in I$ so that $i \leq k$ and $j \leq k$, and

$$\begin{aligned}\mu_i(x_i) + \mu_j(x_j) &= \mu_k(\mu_{ik}(x_i)) + \mu_k(\mu_{jk}(x_j))\\ &= \mu_k(\mu_{ik}(x_i) + \mu_{jk}(x_j)),\end{aligned}$$

so, by definition, for λ, $\nu \in A$, we have

$$\begin{aligned}\alpha(\lambda x + \nu y) &= \alpha_k(\mu_{ik}(\lambda x_i) + \mu_{jk}(\nu x_j))\\ &= \alpha_k(\lambda\mu_{ik}(x_i)) + \alpha_k(\nu\mu_{jk}(x_j))\\ &= \lambda\alpha_k(\mu_{ik}(x_i)) + \nu\alpha_k(\mu_{jk}(x_j))\\ &= \lambda\alpha_i(x_i)) + \nu\alpha_j(x_j))\\ &= \lambda\alpha(x) + \nu\alpha(y).\end{aligned}$$

So, α is a homomorphism of A-modules.

Now, let N, with maps $\alpha_i : M_i \to N$ and $\alpha_i = \alpha_j \circ \mu_{ij}$, satisfy this universal property. and let M be the direct limit, with maps $\mu_i : M_i \to M$ and $\mu_i = \mu_j \circ \mu_{ij}$, which also satisfies this universal property.

By the universal property for M with respect to N, we have a homomorphism $\alpha : M \to N$ satisfying $\alpha_i = \alpha \circ \mu_i$. By the universal property for N with respect to M, we have a homomorphism $\beta : N \to M$ satisfying $\mu_i = \beta \circ \alpha_i$.

Let $x \in M$. Then there exists $x_i \in M_i$ for some $i \in I$ so that $\mu_i(x_i) = x$. Then

$$\beta \circ \alpha(x) = \beta \circ \alpha(\mu_i(x_i)) = \beta \circ \alpha_i(x_i) = \mu_i(x_i) = x.$$

Let $y \in N$. Then there exists $y_j \in M_j$ for some $j \in I$ so that $\alpha_j(y_j) = y$. Then

$$\alpha \circ \beta(y) = \alpha \circ \beta(\alpha_j(y_j)) = \alpha(\mu_j(y_j)) = \alpha_j(y_j)) = y.$$

So, we see that α and β are inverse homomorphisms, so M and N are isomorphic.

$\square$

Exercise 17. Let $(M_i)_{i\in I}$ be a family of submodules of an A-module, such that for each pair of indices i, j in I there exists $k \in I$ such that $M_i + M_j \subseteq M_k$. Define $i \leq j$ to mean that $M_i \subseteq M_j$ and let $\mu_{ij} : M_i \to M_j$ be the embedding of M_i in M_j. Show that

$$\varinjlim M_i = \sum M_i = \bigcup M_i.$$

In particular, any A-module is the direct limit of its finitely generated submodules.

Proof. Let $(M_i)_{i\in I}$ be a family of submodules of an A-module, such that for each pair of indices i, j in I there exists $k \in I$ such that $M_i + M_j \subseteq M_k$. Define $i \leq j$ to mean that $M_i \subseteq M_j$ and let $\mu_{ij} : M_i \to M_j$ be the embedding of M_i in M_j.

Let $M' = \bigcup_{i\in I} M_i$. Then M' is an A-module. Let $\mu_i' : M_i \to M'$ be inclusion. Then $\mu_i' = \mu_j' \circ \mu_{ij}$.

We show M' has the universal property of the direct limit. Let N be any A-module and suppose $\alpha_i : M_i \to N$ be homomorphisms so that $\alpha_i = \alpha_j \circ \mu_{ij}$.

Let $x \in M'$. Then there exists $i \in I$ so that $x = \mu_i'(x_i)$ for some $x_i \in M_i$. Define $\alpha : M' \to N$ by $\alpha(x) = \alpha_i(x_i)$. Suppose $x \in M_j$ as well so that $\mu_j'(x_j) = x$. Then there exists k with i, $j \leq k$ so that $\mu_{ik}(x_i) = \mu_{jk}(x_j)$. Then we have

$$\alpha_i(x_i) = \alpha_k \circ \mu_{ik}(x_i) = \alpha_k \circ \mu_{jk}(x_j) = \alpha_j(x_j),$$

so α is well-defined. Being a composition of homomorphisms, α is a homomorphism, and from its construction it is unique.

Let $M'' = \Sigma_{i\in I} M_i$. Then M'' is an A-module. Let $\mu_i'' : M_i \to M''$ be the inclusion of M_i. Then $\mu_i'' = \mu_j'' \circ \mu_{ij}$.

We show M'' has the universal property of the direct limit. Let N be any A-module and suppose $\alpha_i : M_i \to N$ be homomorphisms so that $\alpha_i = \alpha_j \circ \mu_{ij}$.

Let $x \in M''$. Since x is a (finite) sum of elements of the M_i's, we can use the defining property of a directed set inductively to find one $i \in I$ so that $x \in M_i$. Suppose $x \in M_j$ as well so that $\mu_j''(x) = x$. Then there exists k with i, $j \leq k$ so that $\mu_{ik}(x_i) = \mu_{jk}(x_j)$. Then we have

$$\alpha_i(x) = \alpha_k \circ \mu_{ik}(x) = \alpha_k \circ \mu_{jk}(x) = \alpha_j(x),$$

so α is well-defined. Being a composition of homomorphisms, α is a homomorphism, and from its construction it is unique.

Since $M' = \bigcup_{i\in I} M_i$ and $M'' = \Sigma_{i\in I} M_i$ have the universal property of a direct limit, $\varinjlim M_i = \sum M_i = \bigcup M_i$. □

Exercise 18. Let $\mathbf{M} = (M_i, \mu_{ij})$, $\mathbf{N} = (N_i, \nu_{ij})$ be direct systems of A-modules over the same directed set. Let M, N be the direct limits and $\mu_i : M_i \to M$, $\nu_i : N_i \to N$ the associated homomorphisms. A *homomorphism* $\mathbf{\Phi} : \mathbf{M} \to \mathbf{N}$

is by definition a family of A-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ whenever $i \leq j$. Show that $\mathbf{\Phi}$ defines a unique homomorphism $\phi = \varinjlim \phi_i : M \to N$ such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$.

Proof. Let $\mathbf{M} = (M_i, \mu_{ij})$, $\mathbf{N} = (N_i, \nu_{ij})$ be direct systems of A-modules over the same directed set. Let M, N be the direct limits and $\mu_i : M_i \to M$, $\nu_i : N_i \to N$ the associated homomorphisms.

Let $\mathbf{\Phi} : \mathbf{M} \to \mathbf{N}$ be a homomorphism of directed systems, so that there exists a family of A-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ whenever $i \leq j$. We wish to define a unique homomorphism $\phi = \varinjlim \phi_i : M \to N$ such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$.

Let $x \in M$. By Exercise 15 there exists $x_i \in M_i$ so that $x = \mu_i(x_i)$ for some $i \in I$. By the requirement that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$, we must define

$$\phi(x) = \phi \circ \mu_i(x_i) = \nu_i \circ \phi_i(x_i) = \nu_i(\phi_i(x_i)).$$

This shows ϕ is unique provided it is well-defined.

If $x = \mu_j(x_j)$ for some $j \in I$, $x_j \in M_j$, as well, then there exists $k \in I$ with i, $j \leq k$ so that $\mu_{ik}(x_i) = \mu_{jk}(x_j)$. Then

$$\begin{aligned}
(\nu_i \circ \phi_i)(x_i) &= \nu_i(\phi_i(x_i)) \\
&= (\nu_k \circ \nu_{ik})(\phi_i(x_i)) \\
&= \nu_k((\nu_{ik} \circ \phi_i)(x_i)) \\
&= \nu_k((\phi_k \circ \mu_{ik})(x_i)) \\
&= \nu_k(\phi_k(\mu_{ik}(x_i))) \\
&= \nu_k(\phi_k(\mu_{jk}(x_j))) \\
&= \nu_k((\phi_k \circ \mu_{jk})(x_j)) \\
&= \nu_k((\nu_{jk} \circ \phi_j)(x_j)) \\
&= (\nu_k \circ \nu_{jk})(\phi_j(x_j)) \\
&= \nu_j(\phi_j(x_j)) \\
&= (\nu_j \circ \phi_j)(x_j).
\end{aligned}$$

so ϕ is well-defined.

For x, $y \in M$, we can choose $x_i \in M_i$ and $y_j \in M_j$ so that $\mu_i(x_i) = x$ and $\mu_j(y_j) = y$. Since I is a directed set, there exists $k \in I$ so that i, $j \leq k$. Then

$$\mu_k(\mu_{ik}(x_i)) = (\mu_k \circ \mu_{ik})(x_i) = \mu_i(x_i) = x.$$

Similarly, $\mu_k(\mu_{jk}(y_j)) = y$. So, we may choose a representative for both x and y lying in the same M_k.

For λ, $\mu \in A$ and x, $y \in M$, we can choose x_k, $y_k \in M_k$ so that $\mu_k(x_k) = x$

and $\mu_k(y_k) = y$. Then

$$\begin{aligned}\phi(\lambda x + \mu y) &= \nu_k(\phi_k(\lambda x_k + \mu y_k)) \\ &= \nu_k(\lambda \phi_k(x_k) + \mu \phi_k(\mu y_k)) \\ &= \lambda \nu_k(\phi_k(x_k)) + \mu \nu_k(\phi_k(y_k)) \\ &= \lambda(\nu_k \circ \phi_k)(x_k) + \mu(\nu_k \circ \phi_k)(y_k) \\ &= \lambda \phi(x) + \mu \phi(y).\end{aligned}$$

This proves ϕ is a homomorphism. □

Exercise 19. A sequence of direct systems and homomorphisms

$$\mathbf{M} \to \mathbf{N} \to \mathbf{P}$$

is *exact* if the corresponding sequence of modules and module homomorphisms is exact for each $i \in I$. Show that the sequence $M \to N \to P$ of direct limits is then exact.

Proof. Let

$$\mathbf{M} \xrightarrow{\mathbf{\Phi}} \mathbf{N} \xrightarrow{\mathbf{\Psi}} \mathbf{P}$$

be an exact sequence of direct systems and homomorphisms, so that

$$M_i \xrightarrow{\phi_i} N_i \xrightarrow{\psi_i} P_i$$

is exact for each $i \in I$. We use the notation from Exercise 14 with connecting homomorphisms $\mu_{ij} : M_i \to M_j$, $\nu_{ij} : N_i \to N_j$, and $\rho_{ij} : P_i \to P_j$ whenever $i \leq j$ with the properties that $\mu_j \circ \mu_{ij} = \mu_i$, $\nu_j \circ \nu_{ij} = \nu_i$, and $\rho_j \circ \rho_{ij} = \rho_i$.

By the result of Exercise 18, there exists unique module homomorphism $\phi = \varinjlim \phi_i : M \to N$ satisfying $\phi \circ \mu_i = \nu_i \circ \phi_i$, where $\mu_i : M_i \to M$ and $\nu_i : N_i \to N$, as in Exercise 14. Similarly there exists unique module homomorphism $\psi = \varinjlim \psi_i : N \to P$ satisfying $\psi \circ \nu_i = \rho_i \circ \psi_i$, where $\nu_i : N_i \to N$ and $\rho_i : P_i \to P$.

Suppose $m \in M$ and consider its image $n = \phi(m) \in N$. By the first result in Exercise 15, there exists $m_i \in M_i$ with $\mu_i(m_i) = m$ for some $i \in I$. Then we compute

$$\begin{aligned}
\psi(n) &= \psi(\phi(m)) \\
&= \psi(\phi(\mu_i(m_i))) \\
&= \psi((\phi \circ \mu_i)(m_i)) \\
&= \psi((\nu_i \circ \phi_i)(m_i)) \\
&= (\psi \circ \nu_i)(\phi_i(m_i)) \\
&= (\rho_i \circ \psi_i)(\phi_i(m_i)) \\
&= \rho_i((\psi_i \circ \phi_i)(m_i)) \\
&= \rho_i(0) \\
&= 0,
\end{aligned}$$

since the sequence $M_i \overset{\phi_i}{\to} N_i \overset{\psi_i}{\to} P_i$ is exact. This shows the image of ϕ is contained in the kernel of ψ.

On the other hand, let n be in the kernel of ψ. By the first result in Exercise 15, there exists $n_i \in N_i$ with $\nu_i(n_i) = n$ for some $i \in I$. Then we compute

$$\begin{aligned}
0 &= \psi(n) \\
&= \psi(\nu_i(n_i)) \\
&= (\psi \circ \nu_i)(n_i) \\
&= (\rho_i \circ \psi_i)(n_i) \\
&= \rho_i(\psi_i(n_i)).
\end{aligned}$$

This says that $\psi_i(n_i)$ lies in the kernel of ρ_i.

By second part of Exercise 15, there exists $j \geq i$ so that $\rho_{ij}(\psi_i(n_i)) = 0$. Then we have

$$\begin{aligned}
0 &= \rho_{ij}(\psi_i(n_i)) \\
&= (\rho_{ij} \circ \psi_i)(n_i) \\
&= (\psi_j \circ \nu_{ij})(n_i) \\
&= \psi_j(\nu_{ij}(n_i))
\end{aligned}$$

So, $\nu_{ij}(n_i)$ is in the kernel of ψ_j.

Since the sequence $M_j \overset{\phi_j}{\to} N_j \overset{\psi_j}{\to} P_j$ is exact, there exists $m_j \in M_j$ so that

$\phi_j(m_j) = \nu_{ij}(n_i)$. Let $m = \mu_j(m_j) \in M$. Then

$$\begin{aligned}\phi(m) &= \phi(\mu_j(m_j)) \\ &= (\phi \circ \mu_j)(m_j) \\ &= (\nu_j \circ \phi_j)(m_j) \\ &= \nu_j(\phi_j(m_j)) \\ &= \nu_j(\nu_{ij}(n_i)) \\ &= \nu_i(n_i) \\ &= n,\end{aligned}$$

so the kernel of ψ is contained in the image of ϕ. This shows that the sequence

$$M \to N \to P$$

is exact. $\square$

Mark, start here.

Exercise 20. Keeping the same notation as in Exercise 14, let N be any A-module. Then $(M_i \otimes N, \mu_{ij} \otimes 1)$ is a direct system; let $P = \varinjlim(M_i \otimes N)$ be its direct limit. For each $i \in I$ we have a homomorphism $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$ hence by Exercise 16 a homomorphism $\psi : P \to M \otimes N$. Show that ψ is an isomorphism, so that

$$\varinjlim(M_i \otimes N) \cong (\varinjlim M_i) \otimes N.$$

[For each $i \in I$, let $g_i : M_i \times N \to M_i \otimes N$ be the canonical bilinear mapping. Passing to the limit we obtain a mapping $g : M \times N \to P$. Show that g is A-bilinear and hence define a homomorphism $\phi : M \otimes N \to P$. Verify that $\phi \circ \psi$ and $\psi \circ \phi$ are identity mappings.]

Proof. Keeping the same notation as in Exercise 14, let N be any A-module. Then $(M_i \otimes N, \mu_{ij} \times 1)$ is a direct system; let $P = \varinjlim(M_i \otimes N)$ be its direct limit. For each $i \in I$ we have a homomorphism $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$ hence by Exercise 16 a homomorphism $\psi : P \to M \otimes N$.

Let $g_i : M_i \times N \to M_i \otimes N$ be the canonical bilinear mapping. Passing to the limit we obtain a mapping $g : M \times N \to P$.

$\square$

Exercise 21. Let $(A_i)_{i \in I}$ be a family of rings indexed by a directed set I, and for each pair $i \leq j$ in I let $\alpha_{ij} : A_i \to A_j$ be a ring homomorphism, satisfying condition (1) and (2) of Exercise 14. Regarding each A_i as a $\mathbb{Z}$-module we can then form the direct limit $A = \varinjlim A_i$. Show that A inherits a ring structure from the A_i so that the mappings $A_i \to A$ are ring homomorphisms. The ring A is the *direct limit* of the system (A_i, α_{ij}). If $A = 0$ prove that $A_i = 0$ for some $i \in I$. [Remember that all rings have identity elements!]

Proof. Let $(A_i)_{i \in I}$ be a family of rings indexed by a directed set I, and for each pair $i \leq j$ in I let $\alpha_{ij} : A_i \to A_j$ be a ring homomorphism, satisfying conditions:

i) α_{ii} is the identity mapping of M_i, for all $i \in I$;

ii) $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ whenever $i \leq j \leq k$.

Regarding each A_i as a $\mathbb{Z}$-module, define A to be the direct limit $\varinjlim A_i$.

Define $+$ and $\times$ on $\varinjlim A_i$ as follows. Let x, $y \in \varinjlim A_i$. By Exercise 1, we can choose x_i in A_i so that $\mu_i(x_i) = x$ and y_i in A_i so that $\mu_i(y_i) = y$. By the argument in Exercise 15, we may take these representatives in the same A_i. Now define

$$x + y = \mu_i(x_i + y_i) \qquad xy = \mu_i(x_i y_i),$$

where the elements are the right are elements in $\varinjlim A_i$.

Suppose x_i and x_j represent x and y_i and y_j represent y. Since I is a directed set, there exists $k \in I$ so that i, $j \leq k$. Then $\mu_{ik}(x_i) = x_k$ and $\mu_{jk}(x_j) = x_k$, as well as $\mu_{ik}(y_i) = y_k$ and $\mu_{jk}(y_j) = y_k$. So, we have

$$\begin{aligned}\mu_i(x_i + y_i) &= \mu_i(x_i) + \mu_i(y_i) \\ &= \mu_i(x_i) + \mu_i(y_i) + (\mu_{ik}(\mu_i(x_i)) - \mu_i(x_i)) + (\mu_{ik}(\mu_i(y_i)) - \mu_i(y_i)) \\ &= \mu_{ik}(\mu_i(x_i)) + (\mu_{ik}(\mu_i(y_i))\end{aligned}$$

Mark, finish this.

□

Exercise 22. Let (A_i, α_{ij}) be a direct system of rings and let $\mathfrak{N}_i$ be the nilradical of A_i. Show that $\varinjlim \mathfrak{N}_i$ is the nilradical of $\varinjlim A_i$. If each A_i is an integral domain, then $\varinjlim A_i$ is an integral domain.

Proof. Let (A_i, α_{ij}) be a direct system of rings and let $\mathfrak{N}_i$ be the nilradical of A_i.

□

Exercise 23. Let $(B_\lambda)_{\lambda\in\Lambda}$ be a family of A-algebras. For each finite subset J of Λ let B_J denote the tensor product (over A) of the B_λ for $\lambda \in J$. If J' is another finite subset of Λ and $J \subseteq J'$, there is a canonical A-algebra homomorphism $B_J \to B_{J'}$. Let B denote the direct limit of the rings B_J as J runs through all finite subsets of Λ. The ring B has a natural A-algebra structure for which the homomorphisms $B_J \to B$ are A-algebra homomorphisms. The A-algebra B is the *tensor product* of the family $(B_\lambda)_{\lambda\in\Lambda}$.

Proof. □

Exercise 24. If M is an A-module, the following are equivalent:

i) M is flat;

ii) $\operatorname{Tor}_n^A(M, N) = 0$ for all $n > 0$ and all A-modules N;

iii) $\operatorname{Tor}_1^A(M, N) = 0$ for all A-modules N.

[To show that $(i) \Rightarrow (ii)$, take a free resolution of N and tensor it with M. Since M is flat, the resulting sequence is exact and therefore its homology groups, which are the $\operatorname{Tor}_n^A(M, N) = 0$, are zero for $n > 0$. To show that $(iii) \Rightarrow (i)$, let $0 \to N' \to N \to N'' \to 0$ be an exact sequence. Then, from the Tor exact sequence,

$$\operatorname{Tor}_1(M, N') \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$$

is exact. Since $\operatorname{Tor}_1(M, N') = 0$, it follows that M is flat.]

Proof. □

Exercise 25. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence, with N'' flat. Then N' is flat $\Leftrightarrow$ N is flat.

Proof. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence, with N'' flat. □

Exercise 26. Let N is an A-module. Then N is flat $\Leftrightarrow$ $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$ in A. [Show first that N is flat if $\operatorname{Tor}_1(M, N) = 0$ for all *finitely generated* A-modules M, by using (2.19). If M is finitely generated, let $x_1, \ldots, x_n$ be a set of generators of M, and let M_i be the submodule generated by $x_1, \ldots x_i$. By considering the successive quotients M_i/M_{i-1} and using Exercise 25, deduce that N is flat if $\mathrm{Tor}_1(M, N) = 0$ for all *cyclic* A-modules M, i.e., all M generated by a single element, and therefore of the form $A/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Finally use (2.19) again to reduce to the case where $\mathfrak{a}$ is a finitely generated ideal.]

Proof. □

Exercise 27. A ring A is *absolutely flat* if every A-module is flat. Prove that the following are equivalent:

i) A is absolutely flat.

ii) Every principal ideal is idempotent.

iii) Every finitely generated ideal is a direct summand of A.

[(i)⇒ (ii). Let $x \in A$. Then $A/(x)$ is a flat A-module, hence in the diagram

$$\begin{tikzcd}
(x) \otimes A \arrow[r, "\beta"] \arrow[d] & (x) \otimes A/(x) \arrow[d, "\alpha"] \\
A \arrow[r] & A/(x)
\end{tikzcd}$$

the mapping α is injective. Hence $\mathrm{im}(\beta) = 0$, hence $(x) = (x^2)$. (ii)⇒ (iii). Let $x \in A$. Then $x = ax^2$ for some $a \in A$, hence $e = ax$ is idempotent and we have $(e) = (x)$. Now if e, f are idempotents, then $(e, f) = (e + f - ef)$. Hence every finitely generated ideal is principal, and generated by an idempotent e, hence is a direct summand because $A = (e) \oplus (1 - e)$. (iii)⇒(i). Use the criterion of Exercise 26.

Proof. i)⇒ii) Suppose A is absolutely flat. Let $x \in A$ be nonzero. Let (x) be the principal ideal generated by x. Since A is absolutely flat, the A-module $A/(x)$ is flat. The map $(x) \to A$ is injective, and since $A/(x)$ is a flat A-module, the map $(x) \otimes A/(x) \xrightarrow{\alpha} A \otimes A/(x) = A/(x)$ is also injective by Proposition 2.19. The diagram commutes, and since the composition $(x) \otimes A \to A \to A/(x)$ is the zero map, the composition $(x) \otimes A \to (x) \otimes A/(x) \xrightarrow{\alpha} A/(x)$ must likewise be the zero map. We have just shown that α is injective, so β must be the zero map. It follows that $(x) = (x^2) = (x)^2$, and (x) is idempotent.

ii)⇒iii) Suppose every principal ideal is idempotent. Let $x \in A$. Since $(x) = (x^2)$, $x = ax^2$ for some $a \in A$, hence $e = ax$ is idempotent. If e, f are idempotents, then $(e, f) = (e + f + ef)$. So, every finitely generated ideal is principal, and therefore idempotent, so it's generated by an idempotent e. But then $A = (e) \oplus (1-e)$, so every finitely generated ideal is a direct summand of A.

iii)$\Rightarrow$i) Suppose every finitely generated ideal is a direct summand of A. Let $\mathfrak{a}$ be a finitely generated ideal of A. Then there exists an ideal $\mathfrak{b} \subset A$ with $\mathfrak{a} \oplus \mathfrak{b} = A$.

Let N be any A-module. We have the split exact sequence of A-modules $0 \to \mathfrak{a} \to A \to \mathfrak{b} \to 0$. Tensoring this with N, we get $0 \to \mathfrak{a} \otimes_A N \xrightarrow{\varphi} A \otimes_A N \to A/\mathfrak{a} \otimes_A N \to 0$. By definition, $\mathrm{Tor}_1(A/\mathfrak{a}, N)$ is the kernel of the map φ, so $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$. By Exercise 26, N is flat and since N is arbitrary, A is absolutely flat.

□

Exercise 28. A Boolean ring is absolutely flat. The ring of Chapter 1, Exercise 7 is absolutely flat. Every homomorphic image of an absolutely flat ring is absolutely flat. If a local ring is absolutely flat, then it is a field.

If A is absolutely flat, every non-unit in A is a zero-divisor.

Proof. i) Let A be a Boolean ring. Since $x^2 = x$ for all $x \in A$, it's clear that $(x)^2 = (x^2) = (x)$, so every principal ideal is idempotent. By Exercise 27, A is absolutely flat.

ii) Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$. For $x \in A$, we have $(x)^2 = (x^2) \supseteq (x^n) = (x)$. Since the reverse inclusion is clear, we have $(x)^2 = (x)$. So, every principal ideal is idempotent. By Exercise 27, A is absolutely flat.

iii) Let A be an absolutely flat ring and let $h : A \to B$ be a surjective homomorphism. Let (g) be a principal ideal in B. Since h is surjective, there exists $f \in A$ so that $h(f) = g$. Then the image of (f) under h is (g) since h is surjective. Since A is absolutely flat, the principal ideal (f) is idempotent by Exercise 27. So, $(f^2) = (f)^2 = (f)$. But the image of this under h is $(g^2) = (g)^2 = (g)$, so (g) is idempotent. Since (g) is an arbitrary principal ideal in B, B is absolutely flat, by Exercise 27 again.

iv) Let A be an absolutely flat local ring. Let $x \in \mathfrak{m}$, the unique maximal ideal of A. Since A be an absolutely flat ring, the principal ideal (x) is idempotent by Exercise 27. Then (x) must be generated by an idempotent element. But in a local ring, there are only two idempotents: 0 and 1. Since $x \in \mathfrak{m}$, we must have $x = 0$. This shows that every nonzero element is a unit, so A is a field.

v) Suppose A is absolutely flat and let $x \in A$ be a non-unit. By Exercise 27, every principal ideal is idempotent. Since the ideal (x) is idempotent, there exists $a \in A$ so that $ax^2 = x$. Hence $x(ax - 1) = 0$. If $ax - 1 = 0$, then x is a unit contrary to assumption. So, $ax - 1 \neq 0$, whereby x is a divisor of zero by definition.

Mark, do this last part.

$\square$

Chapter 3

Rings and Modules of Fractions

Exercises

Exercise 1. Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Proof. Let S be a multiplicatively closed subset of a ring A and let M be a finitely generated A-module.

($\Rightarrow$) Suppose $S^{-1}M = 0$. Let $\{m_1, \dots m_n\}$ be a set of generators for M. For each i, $1 \le i \le n$, $m_i/1 \in S^{-1}M = 0$, so there exists $s_i \in S$ so that $m_i s_i = 0$. Let $s = s_1 \cdot \dots \cdot s_n$. Since s annihilates each generator of M, $sM = 0$.

($\Leftarrow$) Suppose there exists $s \in M$ such that $sM = 0$.

Let $m/s' \in S^{-1}M$, where $m \in M$ and $s' \in S$. Since $sm = 0$, we have that $m/s' = 0/s = 0$. Since $m/s' \in S^{-1}M$ is arbitrary, $S^{-1}M = 0$. □ □

Exercise 2. Let $\mathfrak{a}$ be an ideal of a ring A, and let $S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Use this result and Nakayama's lemma to give a proof which does not depend on determinants of the following fact:

> Let M be a finitely generated A-module and let $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a}M = M$. Then there exists $x \equiv 1 (\operatorname{mod} \mathfrak{a})$ such that $xM = 0$.

[If $M = \mathfrak{a}M$, then $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$, and hence by Nakayama's Lemma we have $S^{-1}M = 0$.]

Proof. Let $\mathfrak{a}$ be an ideal of a ring A, and let $S = 1 + \mathfrak{a}$.

Let $x/(1+a) \in S^{-1}\mathfrak{a}$, with x, $a \in \mathfrak{a}$. Let $y/(1+b) \in S^{-1}A$, with $y \in A$ and $b \in \mathfrak{a}$ be arbitrary. Then

$$\begin{aligned} 1 - \left(\frac{x}{1+a}\right)\left(\frac{y}{1+b}\right) &= \frac{(1+a)(1+b) - xy}{(1+a)(1+b)} \\ &= \frac{1+a+b+ab-xy}{(1+a)(1+b)}. \end{aligned}$$

Since a, b, and x are in the ideal $\mathfrak{a}$, we see the numerator here is an element of $1 + \mathfrak{a}$, so this element is a unit in $S^{-1}A$. By Proposition 1.9 in Chapter 1, the element $x/(1+a)$ lies in the Jacobson radical of $S^{-1}A$. Since $x/(1+a) \in S^{-1}\mathfrak{a}$ is arbitrary, $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Now, suppose M is a finitely generated A-module and suppose $\mathfrak{a}$ is an ideal of A such that $\mathfrak{a}M = M$. Note that

$$\begin{aligned} \mathfrak{a}M &= M \\ S^{-1}(\mathfrak{a}M) &= S^{-1}M \\ S^{-1}(\mathfrak{a})S^{-1}M &= S^{-1}M. \end{aligned}$$

Since $S^{-1}(\mathfrak{a})$ is contained in the Jacobson radical of $S^{-1}A$ and $S^{-1}M$ is finitely generated, we may apply Nakayama's Lemma to obtain

$$S^{-1}M = 0.$$

By Exercise 1 of this section, we have that there exists $x \in S$ so that $xM = 0$. Since $x \in S = 1 + \mathfrak{a}$, we have that $x \equiv 1 (\mathrm{mod}\, \mathfrak{a})$, as desired. □

Exercise 3. Let A be a ring, let S and T be two multiplicatively closed subsets of A, and let U be the image of T in $S^{-1}A$. Show that the rings $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic.

Proof. Let A be a ring, let S and T be two multiplicatively closed subsets of A, and let U be the image of T in $S^{-1}A$. Define

$$\begin{aligned} \varphi &: A \to (ST)^{-1}A \\ \varphi(a) &\mapsto \frac{a}{1} \end{aligned}$$

For $s \in S \subset A$, $\varphi(s)$ is a unit in $(ST)^{-1}A$, so φ induces a homomorphism

$$\begin{aligned} \varphi' &: S^{-1}A \to (ST)^{-1}A \\ \varphi'(a/s) &\mapsto \frac{a/s}{1} = \frac{a}{s}. \end{aligned}$$

For $t/1 \in U$,

$$\varphi'(t/1) = \frac{t/1}{1} = \frac{t}{1},$$

which is a unit in $(ST)^{-1}A$, so φ' induces a homomorphism

$$\varphi'' : U^{-1}(S^{-1}A) \to (ST)^{-1}A$$
$$\varphi''\left(\frac{a/s}{t/s'}\right) \mapsto \frac{as'}{st}.$$

The map φ'' is certainly surjective. Suppose $\varphi''\left(\frac{a/s}{t/s'}\right) = 0$. Then

$$\varphi''\left(\frac{a/s}{t/s'}\right) = \frac{as'}{st} = 0.$$

This happens if and only if some element of $s_0t_0 \in ST$ satisfies $s_0t_0as' = 0$. But then

$$\frac{a/s}{t/s'} = \frac{as'}{st} = \frac{as's_0t_0}{sts_0t_0} = \frac{0}{sts_0t_0} = 0.$$

So, we see φ'' is injective as well. So, the rings $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic. ☐

Exercise 4. Let $f : A \to B$ be a homomorphism of rings and let S be a multiplicatively closed subset of A. Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

Proof. Let $f : A \to B$ be a homomorphism of rings and let S be a multiplicatively closed subset of A. We remark that the map f makes B an A-module by defining $a \cdot b$ to mean $f(a)b \in B$. Since S is a multiplicatively closed subset of A, we have $s \cdot b$ to mean $f(s)b \in B$.

Let $T = f(S)$. Since S is multiplicatively closed and f is a homomorphism, T is also multiplicatively closed. Define

$$\varphi : S^{-1}B \to T^{-1}B$$
$$\varphi\left(\frac{b}{s}\right) \mapsto \frac{b}{f(s)}.$$

Let b/s, $b'/s' \in S^{-1}B$. Then

$$\varphi\left(\frac{b}{s} + \frac{b'}{s'}\right) = \varphi\left(\frac{s'b + sb'}{ss'}\right)$$
$$= \frac{s'b + sb'}{f(ss')}.$$

By the way multiplication is defined between A and B, this equals

$$\begin{aligned}\frac{f(s')b + f(s)b'}{f(ss')} &= \frac{f(s')b + f(s)b'}{f(s)f(s')} \\ &= \frac{b}{f(s)} + \frac{b'}{f(s')} \\ &= \varphi\left(\frac{b}{s}\right) + \varphi\left(\frac{b}{s'}\right).\end{aligned}$$

Let $a/s \in S^{-1}A$ and $b/s' \in S^{-1}B$. Then

$$\begin{aligned}\varphi\left(\frac{a}{s} \cdot \frac{b}{s'}\right) &= \varphi\left(\frac{ab}{ss'}\right) \\ &= \frac{ab}{f(ss')} \\ &= \frac{ab}{f(s)f(s')}.\end{aligned}$$

By the way multiplication is defined between A and B, this equals

$$\begin{aligned}\frac{f(a)b}{f(s)f(s')} &= \frac{f(a)}{f(s)}\frac{b}{f(s')} \\ &= f\left(\frac{a}{s}\right)\frac{b}{f(s')}\end{aligned}$$

Once again, by the way multiplication is defined, this equals

$$\frac{a}{s} \cdot \frac{b}{f(s')} = \frac{a}{s} \cdot \varphi\left(\frac{b}{s'}\right).$$

So, φ is a homomorphism of $S^{-1}A$ modules.

Suppose $\varphi(b/s) = 0$. Then $b/f(s) = 0$ in $T^{-1}(B)$, which means that there exists $f(s') \in f(S) = T$ so that $bf(s') = 0$. By the way multiplication is defined, this means $b \cdot s' = 0$, which shows $b/s = 0$ in $S^{-1}B$. So, φ is injective.

This map is easily seen to be surjective, so φ is an isomorphism of $S^{-1}A$-modules. $\square$

Exercise 5. Let A be a ring. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each $A_\mathfrak{p}$ is an integral domain, is A necessarily an integral domain?

Proof. Let A be a ring. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$.

Let $x \in A$ be nilpotent. Assume $x \neq 0$ and let $\mathfrak{a}$ be the annihilator of A. Since every (proper) ideal is contained in a maximal ideal, let $\mathfrak{p}$ be a maximal ideal containing $\mathfrak{a}$. Since $x \in A$ is nilpotent, $x/1 \in A_\mathfrak{p}$ is also nilpotent. Since $A_\mathfrak{p}$ has no nilpotent element unequal to zero, we must have $x/1 = 0$ in $A_\mathfrak{p}$. Hence, there exists some $m \notin \mathfrak{p}$ such that $mx = 0$, but this contradicts the fact that $\mathfrak{p}$ contains the annihilator of x. Thus, $x \in A$ must be zero, whereby A does not contain any nilpotents unequal to zero. $\square$

For the second part, let $A = \mathbb{Z}/6\mathbb{Z}$, the ring of integers modulo 6. This ring has two prime ideals: (2) and (3).

I claim that $A_{(2)}$ is an integral domain. From Proposition 3.3, is follows that

$$(\mathbb{Z}/6\mathbb{Z})_{(2)} \cong \mathbb{Z}_{(2)}/6\mathbb{Z}_{(2)}.$$

But $\mathbb{Z}_{(2)}/6\mathbb{Z}_{(2)} = \mathbb{Z}_{(2)}/2\mathbb{Z}_{(2)}$ since 3 is a unit in $\mathbb{Z}_{(2)}$. So, we have

$$(\mathbb{Z}/6\mathbb{Z})_{(2)} \cong \mathbb{Z}_{(2)}/6\mathbb{Z}_{(2)} \cong \mathbb{Z}_{(2)}/2\mathbb{Z}_{(2)} \cong (\mathbb{Z}/2\mathbb{Z})_{(2)}$$

from Proposition 3.3 again. However, $\mathbb{Z}/2\mathbb{Z}$ is a field and its localization $(\mathbb{Z}/2\mathbb{Z})_{(2)}$ is that same field. So, we have $A_{(2)} = (\mathbb{Z}/6\mathbb{Z})_{(2)}$ is a field, so it's certainly an integral domain.

Similarly $A_{(3)}$ is also an integral domain.

So, here's an example of a ring A so that the localization $A_\mathfrak{p}$ at each prime ideal $\mathfrak{p} \subseteq A$ is a domain, but A itself is not a domain.

Exercise 6. Let A be a ring $\neq 0$ and let Σ be the set of all multiplicatively closed subsets S of A such that $0 \notin S$. Show that Σ has maximal elements, and that $S \in \Sigma$ is maximal if and only if $A \setminus S$ is a minimal prime ideal of A.

Proof. Let A be a ring $\neq 0$ and let Σ be the set of all multiplicatively closed subsets S of A such that $0 \notin S$.

We apply Zorn's Lemma to show that Σ contains maximal elements. Let $S_1 \subset S_2 \subset S_3 \subset \cdots$ be a chain in Σ. Let $S = \cup_i S_i$, so that certainly $S_i \subset S$ for all i. Suppose $x,\ y \in S$. Then $x \in S_i$ and $y \in S_j$ for some $i,\ j$. Let $k = \max\{i, j\}$. Since the S_ℓ's are a nested family of sets, $x,\ y \in S_k$. Since S_k is multiplicatively closed, $xy \in S_k \subset S$. Hence, S is multiplicatively closed. Since $0 \notin S_i$ for all i, $0 \notin S$. Hence, $S \in \Sigma$. By Zorn's Lemma, Σ contains maximal elements.

Let S be a maximal element in Σ. Since $0 \notin S$, $S^{-1}A$ is not the zero ring. Let $\mathfrak{m}$ be any maximal ideal in $S^{-1}A$ and let $\mathfrak{p}$ be its contraction in A. Note that $\mathfrak{p}$ must be a prime ideal. Since $\mathfrak{p}$ is the contraction of an ideal in $S^{-1}A$, $S \cap \mathfrak{p} = \emptyset$, so $S \subset A - \mathfrak{p}$. However, $A - \mathfrak{p}$ is multiplicatively closed since $\mathfrak{p}$ is a prime ideal, by maximality, $S = A - \mathfrak{p}$.

Suppose $\mathfrak{q} \subsetneq \mathfrak{p}$ is likewise a prime ideal. Then $S' = A \setminus \mathfrak{q}$ is a multiplicatively closed set which does not contain 0, so $S' \in \Sigma$. Since $\mathfrak{q} \subsetneq \mathfrak{p}$, we have $S' \supsetneq S$, which contradicts the maximality of S in Σ. Thus, $\mathfrak{p}$ is a minimal prime ideal of A. It is easily seen that this argument is reversible, so that if $\mathfrak{p}$ is a minimal prime ideal of A, then S is a maximal multiplicatively closed subset of A not containing 0. $\square$

Exercise 7. A multiplicatively closed subset S of a ring A is said to be *saturated* if

$$xy \in S \iff x \in S \text{ and } y \in S.$$

Prove that

i) S is saturated $\iff A \setminus S$ is a union of prime ideals.

ii) If S is any multiplicatively closed subset of A, there is a unique smallest saturated multiplicatively closed subset $\overline{S}$ containing S, and that $\overline{S}$ is the complement in A of the union of the prime ideals which do not meet S. ($\overline{S}$ is called the *saturation* of S.)

If $S = 1 + \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of A, find $\overline{S}$.

i) *Proof.* Let A be a ring and let $S \subset A$ be a multiplicatively closed subset.

   ($\Rightarrow$) Suppose S is a saturated multiplicatively closed set in A. We need to show that every element $x \in A \setminus S$ lies in a prime ideal $\mathfrak{p}$ disjoint from S.

   Let $x \in A \setminus S$. Since S is saturated, $(x) \cap S = \emptyset$. So, $(x)^e \neq (1)$ in $S^{-1}A$ by Proposition 3.3.11(ii). So, x is a not a unit in $S^{-1}A$. Since $(x) \cap S = \emptyset$, we have $0 \notin S$, so $S^{-1}A$ is not the zero ring. Let $\mathfrak{m}$ be any maximal ideal in $S^{-1}A$ containing $(x)^e$. Note that $\mathfrak{m} \cap S = \emptyset$. Let $\mathfrak{p} = \mathfrak{m}^c = A \cap \mathfrak{m}$. Then $\mathfrak{p}$ is a prime ideal containing x and disjoint from S.

   ($\Leftarrow$) Suppose $A \setminus S = \cup_i \mathfrak{p}_i$ is a union of prime ideals.

   Suppose $xy \in S$. Then $xy \notin \cup_i \mathfrak{p}_i$. So, for all $i \in I$, $xy \notin \mathfrak{p}_i$, and since $\mathfrak{p}_i$ is an ideal, $x \notin \mathfrak{p}_i$ and $y \notin \mathfrak{p}_i$. So, $x \notin \cup_i \mathfrak{p}_i$ and $y \notin \cup_i \mathfrak{p}_i$. That is, $x \in S$ and $y \in S$.

   Suppose $x \in S$ and $y \in S$. Then $x \notin \cup_i \mathfrak{p}_i$ and $y \notin \cup_i \mathfrak{p}_i$. Then $x, y \notin \mathfrak{p}_i$ for all $i \in I$. Since $\mathfrak{p}_i$ is a prime ideal, $xy \notin \mathfrak{p}_i$ for all $i \in I$. That is, $xy \notin \cup_i \mathfrak{p}_i$, whereby $xy \in S$.

   This shows S is saturated. $\square$

ii) *Proof.* Let Σ be the collection of all saturated sets containing S. This set is nonempty since A itself is an element. Let $\overline{S}$ be the intersection of all elements of Σ. We show $\overline{S}$ is saturated. Suppose $xy \in \overline{S}$. Then $xy \in S'$ for all $S' \in \Sigma$. Since S' is saturated, $x \in S'$ and $y \in S'$. Since this is true for all $S' \in \Sigma$, $x \in \overline{S}$ and $y \in \overline{S}$. Conversely, suppose $x \in \overline{S}$ and $y \in \overline{S}$. Then x and y are in S' for all $S' \in \Sigma$. Since S' is multiplicatively closed, $xy \in S' \in \Sigma$ for all $S' \in \Sigma$. That is, $xy \in \overline{S}$. This shows $\overline{S}$ is saturated. Clearly it's the smallest saturated set containing S (since it's the intersection of all of them).

By part (i) of this problem, since $\overline{S}$ is saturated, $A \setminus \overline{S}$ is a union of prime ideals disjoint from $\overline{S}$.

For notation, let

$$\overline{S} = \bigcap_{S' \in \Sigma} S'.$$

For each saturated set $S' \in \Sigma$, by part (i), the set $A \setminus S'$ is a union of prime ideals disjoint from S'. Since $S' \supseteq S$, these prime ideals are disjoint from S. Say $A \setminus S' = \cup_\alpha \mathfrak{p}_\alpha$ where $\mathfrak{p}_\alpha$ is a prime ideal disjoint from S. Then

$$\begin{aligned} A \setminus \overline{S} &= A \setminus \bigcap_{S' \in \Sigma} S' \\ &= \bigcup_{S' \in \Sigma} A \setminus S' \\ &= \bigcup_{S' \in \Sigma} \cup_\alpha \mathfrak{p}_\alpha. \end{aligned}$$

Is every prime ideal not meeting S in this union? Let $\mathfrak{p}$ be a prime ideal disjoint from S. Then $S' := A \setminus \mathfrak{p}$ is saturated and contains S, so $S' \in \Sigma$. It follows from the definition of $\overline{S}$ that $\mathfrak{p}$ is in this union. So, we see that the complement of $\overline{S}$ is the union of all prime ideals in A not meeting S. $\square$

Let $S = 1 + \mathfrak{a}$. We know from (ii) that $A \setminus \overline{S}$ is the union of all prime ideals disjoint from S.

Suppose $\mathfrak{p}$ is a prime ideal meeting S. Then there exist $x \in \mathfrak{p}$ and $a \in \mathfrak{a}$ so that $x = 1 + a$. That is, $1 \in \mathfrak{p} + \mathfrak{a}$. Conversely, if $1 \in \mathfrak{p} + \mathfrak{a}$, then there exist $x \in \mathfrak{p}$ and $a \in \mathfrak{a}$ so that $x = 1 + a$. That is, $\mathfrak{p}$ is a prime ideal meeting S.

So, we have

$$\overline{S} = \bigcup_{\mathfrak{p}: 1 \notin \mathfrak{p} + \mathfrak{a}} \mathfrak{p}$$

Which prime ideals are these?

If $\mathfrak{m}$ is a maximal ideal containing $\mathfrak{a}$, then $1 \notin \mathfrak{m} + \mathfrak{a}$, so $\mathfrak{m}$ is one of the prime ideals in this union.

Conversely, if $\mathfrak{p}$ is a prime ideal with $1 \notin \mathfrak{p} + \mathfrak{a}$, then there exists a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p} + \mathfrak{a}$, and this implies that $\mathfrak{a} \subset \mathfrak{m}$. It follows that $1 \notin \mathfrak{m} + \mathfrak{a}$.

So, we see that

$$\overline{S} = A \setminus \bigcup_{\mathfrak{m}: \mathfrak{m} \supseteq \mathfrak{a}} \mathfrak{m}.$$

Exercise 8. Let S, T be a multiplicatively closed subsets of A, such that $S \subseteq T$. Let $\varphi : S^{-1}A \to T^{-1}A$ be the homomorphism which maps each $a/s \in S^{-1}A$ to a/s considered as an element of $T^{-1}A$. Show that the following statements are equivalent:

i) φ is bijective.

ii) For each $t \in T$, $t/1$ is a unit in $S^{-1}A$.

iii) For each $t \in T$ there exists $x \in A$ such that $xt \in S$.

iv) T is contained in the saturation of S.

v) Every prime ideal which meets T also meets S.

Proof. Let S, T be a multiplicatively closed subsets of A, such that $S \subseteq T$. Let $\varphi : S^{-1}A \to T^{-1}A$ be the homomorphism which maps each $a/s \in S^{-1}A$ to a/s considered as an element of $T^{-1}A$.

i) $\Rightarrow$ ii) Suppose φ is bijective. For $t \in T$, $\varphi(t/1) = t/1$ is a unit in $T^{-1}A$. So there exists a multiplicative inverse $v \in T^{-1}A$. Since φ is bijective, there exists $a/s \in S^{-1}A$ with $\varphi(a/s) = v$. Then we have $\varphi(a/s \cdot t/1) = \varphi(a/s)\varphi(t/1) = v \cdot t/1 = 1$. Since φ is bijective and $\varphi(1)$ is also 1, we must have $a/s \cdot t/1 = 1$ in $S^{-1}A$. This makes $t/1$ a unit in $S^{-1}A$.

ii) $\Rightarrow$ iii) Suppose for each $t \in T$, $t/1$ is a unit in $S^{-1}A$. Let $t \in T$. By hypothesis, $t/1$ is a unit in $S^{-1}A$, so there exists $a/s \in S^{-1}A$ so that $(a/s) \cdot (t/1) = at/s = 1 \in S^{-1}A$. This means there exists $s' \in S$ so that $s'(at - s) = 0$. So, we see that $s'at = s's \in S$. Setting $x = s'a \in A$, we have $xt \in S$.

iii) $\Rightarrow$ iv) Suppose for each $t \in T$ there exists $x \in A$ such that $xt \in S$.

Let $t \in T$. Suppose $t \notin \overline{S}$. By Exercise 7(ii), $t \in \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal in A not meeting S. By hypothesis, there exists $x \in A$ so that $xt \in S$. But then $xt \in \mathfrak{p} \cap S$, a contradiction. So $t \in \overline{S}$. Since $t \in T$ is arbitrary, $T \subseteq \overline{S}$.

iv) $\Rightarrow$ v) Suppose $T \subseteq \overline{S}$. Let $\mathfrak{p} \subset A$ be a prime ideal which meets T. Let $t \in T \cap \mathfrak{p}$. Recall $A \setminus \overline{S}$ is the union of all prime ideals not meeting S. Since $t \in T \subseteq \overline{S}$, t does not lie in any prime ideal not meeting S. It follows that $\mathfrak{p}$ meets S.

v) $\Rightarrow$ iii) Suppose every prime ideal which meets T also meets S.

Suppose (iii) is false. Then for some $t \in T$, we have $(t) \cap S = \emptyset$. Since $(t) \cap S = \emptyset$, $t/1$ is not a unit in $S^{-1}A$, so there exists a maximal ideal $\mathfrak{m}$ in $S^{-1}A$ with $t/1 \in \mathfrak{m}$. Then t is an element of the prime ideal $\mathfrak{p} = A \cap \mathfrak{m}$. By hypothesis, we must have that $\mathfrak{p}$ meets S. But if $\mathfrak{p}$ meets S, then $\mathfrak{m}$ meets S and $\mathfrak{m} = (1)$ in $S^{-1}A$, a contradiction. So, (iii) is true.

iii) $\Rightarrow$ ii) Suppose for every $t \in T$ there exists $x \in A$ so that $xt \in S$.

Let $t \in T$. By hypothesis, there exists $x \in A$ so that $xt = s \in S$. Then in $S^{-1}A$, we have $(t/1)(x/s) = xt/s = 1$, so t is a unit in $S^{-1}A$. Since $t \in T$ is arbitrary, every element of T is a unit in $S^{-1}A$.

ii) $\Rightarrow$ i) Suppose every element of T is a unit in $S^{-1}A$. Let $\varphi : S^{-1}A \to T^{-1}A$ be the homomorphism which maps each $a/s \in S^{-1}A$ to a/s considered as an element of $T^{-1}A$. Define $\psi : T^{-1}A \to S^{-1}A$ by $\psi(a/t) = a/t$. Since every element of $t \in T$ is a unit in $S^{-1}A$, this makes sense.

Since $\varphi \circ \psi = \mathrm{id}_{T^{-1}A}$ and $\psi \circ \varphi = \mathrm{id}_{S^{-1}A}$, we see that φ is a bijection.

□

Exercise 9. The set S_0 of all non-zero-divisors in A is a saturated multiplicatively closed subset of A. Hence the set D of zero-divisors in A is a union of prime ideals. Show that every minimal prime ideal of A is contained in D. (Use Exercise 6.)

The ring $S_0^{-1}A$ is called the *total ring of fractions* of A. Prove that

i) S_0 is the largest multiplicatively closed subset of A for which the homomorphism $A \to S_0^{-1}A$ is injective.

ii) Every element in $S_0^{-1}A$ is either a zero-divisor or a unit.

iii) Every ring in which every non-unit is a zero-divisor is equal to its total ring of fractions (i.e., $A \to S_0^{-1}A$ is bijective).

Proof. Let S_0 be the set of all non-zero-divisors in A. If $x, y \in A$, with $xy \in S_0$, it's easily seen that $x, y \in S_0$. Conversely, let $x, y \in S_0$. If xy is zero-divisor, there exists a nonzero $a \in A$ so that $axy = 0$. Since x is not a divisor of zero, $ax \neq 0$. But then it follows that y is a zero-divisor, contrary to assumption. So, $xy \in S_0$ and S_0 is a multiplicatively closed subset of A. These two implications show that S_0 is saturated. By Exercise 7(i), the set of zero divisors, $D = A \setminus S_0$, is a union of prime ideals.

Let $\mathfrak{p}$ be a prime ideal in A. Suppose $x \in \mathfrak{p}$ is not a zero divisor. Consider the set $T = \{x^i y \,|\, y \in A - \mathfrak{p}\}$. Then T is multiplicatively closed and, since x is not a zero divisor, T doesn't contain zero. Further T properly contains $A \setminus \mathfrak{p}$. By Exercise 6, $\mathfrak{p}$ is not a minimal prime ideal. So, any prime ideal not contained in D is not a minimal prime ideal. Equivalently, any minimal prime ideal is contained in D.

i) Let S be a multiplicatively closed set in A and let $\varphi : A \to S^{-1}A$ be the natural homomorphism. If a lies in the kernel of φ, then $a/1 = 0$ in $S^{-1}A$. It follows that there exists $s \in S$ so that $as = 0$. So, $a = 0$ or s is a zero divisor. Conversely, if s is a zero divisor, then there exist a nonzero $a \in A$ so that $as = 0$. It follows that a lies in the kernel of φ.

Consequently, if φ is injective, S cannot contain any nonzero zero divisors. Thus $S \subseteq S_0$. Thus, S_0 is the largest multiplicatively closed subset so that $\varphi : A \to S_0^{-1}A$ is injective.

ii) Let a/s be an element of $S_0^{-1}A$. If a is a non-zero divisor, then $a \in S_0$ and a/s is a unit in $S_0^{-1}A$. If a is a zero-divisor, there exists $b \in A$, $b \neq 0$, so that $ab = 0$ But then $a/s \cdot b/1 = 0$ in $S_0^{-1}A$. If $b/1 = 0$ in $S_0^{-1}A$, then there exists $t \in S_0$ so that $tb = 0$, but since $b \neq 0$, this makes t a divisor of zero, which contradicts the fact that $t \in S_0$. So, $a/s \cdot b/1 = 0$ and $b/1 \neq 0$ in $S_0^{-1}A$, so a/s is a zero-divisor in $S_0^{-1}A$.

iii) Suppose A is a ring in which every non-unit is a zero-divisor. Consider the natural map $\varphi : A \to S_0^{-1}A$. Suppose $\varphi(a) = 0$. Then there exists $s \in S_0$ so that $as = 0$. Since $s \in S_0$, s is not a zero-divisor, so this forces $a = 0$. Hence φ is injective. Let $a/s \in S_0^{-1}A$. Since $s \in S_0$, s is a non-zero-divisor, so it follows that s is a unit in A. Then $\phi(as^{-1}) = as^{-1}/1 = a/s$, so φ is surjective. Hence, φ is an isomorphism.

□

Exercise 10. Let A be a ring.

i) If A is absolutely flat, and S is any multiplicatively closed subset of A, then $S^{-1}A$ is absolutely flat.

ii) A is absolutely flat $\Leftrightarrow$ $A_{\mathfrak{m}}$ is a field for each maximal ideal $\mathfrak{m}$.

Proof. Let A be a ring.

i) Suppose A is absolutely flat and let S be any multiplicatively closed subset of A.

Let M be any $S^{-1}A$-module and let $0 \to P \to Q \to R \to 0$ be an exact sequence of $S^{-1}A$-modules.

We can consider M to be an A-module and $0 \to P \to Q \to R \to 0$ an exact sequence of A-modules by restriction of scalars. Since A is absolutely flat, as have

$$0 \to P \otimes_A M \to Q \otimes_A M \to R \otimes_A M \to 0$$

is exact. By Proposition 3.3, we have

$$0 \to S^{-1}(P \otimes_A M) \to S^{-1}(Q \otimes_A M) \to S^{-1}(R \otimes_A M) \to 0$$

is exact, and then by Proposition 3.7, we have

$$0 \to S^{-1}P \otimes_{S^{-1}A} S^{-1}M \to S^{-1}Q \otimes_{S^{-1}A} S^{-1}M \to S^{-1}R \otimes_{S^{-1}A} S^{-1}M \to 0$$

is also exact.

So, we've reduced the problem to showing that if M is an $S^{-1}A$-module and M_A is M considered as an A-module by restriction of scalars, then $S^{-1}(M_A) = M$.

Define $\varphi : M \to S^{-1}M_A$ by $\varphi(m) = m$. Suppose $\varphi(m) = 0$ in $S^{-1}M_A$. Then there exists $s \in S$ so that $sm = 0$. In $S^{-1}A$, s is a unit, so this last equation implies that $m = 0$. So, φ is injective. Let $m/s \in S^{-1}M_A$. Since M is a $S^{-1}A$-module, $(1/s)m \in M$. Then $\varphi((1/s)m) = m/s$, so φ is surjective. Hence, φ is an isomorphism.

It now follows that $S^{-1}A$ is absolutely flat.

ii) ($\Rightarrow$) Suppose A is absolutely flat and let $\mathfrak{m}$ be a maximal ideal in A. By part (i), $A_\mathfrak{m}$ is an absolutely flat local ring. By Exercise 28 in Chapter 2, $A_\mathfrak{m}$ is a field.

($\Leftarrow$) Suppose $A_\mathfrak{m}$ is a field for each maximal ideal $\mathfrak{m}$ in A. Let M be an A-module and let $\mathfrak{m}$ be a maximal ideal in A. Then $M_\mathfrak{m}$ is a $A_\mathfrak{m}$-module—a $A_\mathfrak{m}$ vector space. A vector space is a free module, so it's a direct sum of copies of $A_\mathfrak{m}$, so by Exercise 4 in Chapter 2, $M_\mathfrak{m}$ is a flat $A_\mathfrak{m}$-module. Since $\mathfrak{m}$ is an arbitrary maximal ideal in A, M is a flat A-module by Proposition 3.10.

□

Exercise 11. Let A be a ring. Prove that the following are equivalent:

i) $A/\mathfrak{N}$ is absolutely flat ($\mathfrak{N}$ being the nilradical of A).

ii) Every prime ideal of A is maximal.

iii) Spec(A) is a T_1-space (i.e., every subset consisting of a single point is closed).

iv) Spec(A) is Hausdorff.

If these conditions are satisfied, show that Spec(A) is compact and totally disconnected (i.e., the only connected subsets of Spec(A) are those consisting of a single point).

Proof. Let A be a ring and let $\mathfrak{N}$ be the nilradical of A.

i) $\Rightarrow$ ii) Let $A/\mathfrak{N}$ be absolutely flat. Let $\mathfrak{p}$ be a prime ideal in A. Then $\mathfrak{p}$ is contained in a maximal ideal $\mathfrak{m}$ by Corollary 1.4 in Chapter 1. Consider the inclusion

$$0 \to \mathfrak{p} \to \mathfrak{m}.$$

Tensoring with $A/\mathfrak{m}$ and noting that $A/\mathfrak{m}$ is an $A/\mathfrak{N}$ module, and therefore flat, we get

$$0 \to \mathfrak{p} \otimes A/\mathfrak{m} \to \mathfrak{m} \otimes A/\mathfrak{m} = 0.$$

So, $\mathfrak{p} \otimes A/\mathfrak{m} = 0$. The module $\mathfrak{p} \otimes A/\mathfrak{m}$ is isomorphic to $\mathfrak{p}/\mathfrak{m}$, so $\mathfrak{p} = \mathfrak{m}$ and $\mathfrak{p}$ is a maximal ideal.

ii) $\Leftrightarrow$ iii) A point $x_{\mathfrak{p}}$ is closed in $\operatorname{Spec}(A)$ if and only if the only prime ideal containing $\mathfrak{p}$ is $\mathfrak{p}$, that is, $\mathfrak{p}$ is a maximal ideal.

iii) $\Rightarrow$ iv) Suppose $\operatorname{Spec}(A)$ is a T_1-space. Then each point of $\operatorname{Spec}(A)$ is closed and therefore every prime ideal in A is a maximal ideal.

Let $x_{\mathfrak{p}}$ and $x_{\mathfrak{q}}$ be distinct points of $\operatorname{Spec}(A)$ corresponding to distinct prime (and therefore maximal) ideals $\mathfrak{p}$ and $\mathfrak{q}$ in A. Since $\mathfrak{p} \neq \mathfrak{q}$, there exists $f \in \mathfrak{p}$ with $f \notin \mathfrak{q}$. The ring $A_{\mathfrak{p}}$ has exactly one prime ideal, $\mathfrak{p}A_{\mathfrak{p}}$, which coincides with its nilradical. So, $f/1$ is nilpotent in $A_{\mathfrak{p}}$, whereby there exists $n \in \mathbb{N}$ so that $f^n/1 = 0$ in $A_{\mathfrak{p}}$. That is, there exists $s \in A \backslash \mathfrak{p}$ so that $f^n s = 0$. Let $D(s) = \operatorname{Spec}(A) \setminus V(s)$ and let $D(f) = \operatorname{Spec}(A) \setminus V(f)$. Then $D(s)$ and $D(f)$ are disjoint open sets in $\operatorname{Spec}(A)$, $x_{\mathfrak{p}} \in D(s)$, and $x_{\mathfrak{q}} \in D(f)$. So, $\operatorname{Spec}(A)$ is Hausdorff.

iv) $\Rightarrow$ i) Suppose $\operatorname{Spec}(A)$ is a Hausdorff space. Then $\operatorname{Spec}(A)$ is a T_1-space, whereby every point is closed, whereby every prime ideal is maximal.

Let $\mathfrak{N}$ be the nilradical of A and consider $A/\mathfrak{N}$. Let $\mathfrak{m}$ be a maximal ideal in $A/\mathfrak{N}$. Then $\mathfrak{m}$ corresponds to a maximal ideal $\mathfrak{m}'$ in A which contains $\mathfrak{N}$. Let $\mathfrak{p}$ be any prime ideal contained between $\mathfrak{N}$ and $\mathfrak{m}'$. Since every prime ideal is maximal, we must have that $\mathfrak{p} = \mathfrak{m}'$. It follows that $(A/\mathfrak{N})_{\mathfrak{m}} = A_{\mathfrak{m}}/\mathfrak{N}_{\mathfrak{m}}$ contains only one maximal ideal, the 0 ideal. Hence $(A/\mathfrak{N})_{\mathfrak{m}}$ is a field, whereby $A/\mathfrak{N}$ is absolutely flat, by Problem 10.

$\square$

Exercise 12. Let A be an integral domain and M an A-module. An element $x \in M$ is a *torsion element* of M if $\operatorname{Ann}(x) \neq 0$, that is if x is killed by some non-zero element of A. Show that the torsion elements of M form a submodule of M. This submodule is called the *torsion submodule* of M and is denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

i) If M is any A-module, then $M/T(M)$ is torsion-free.

ii) If $f : M \to N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.

iii) If $0 \to M' \to M \to M''$ is an exact sequence, then the sequence $0 \to T(M') \to T(M) \to T(M'')$ is exact.

iv) If M is any A-module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of M into $K \otimes_A M$, where K is the field of fractions of A.

[For (iv), show that K may be regarded as the direct limit of its submodules $A\xi$ ($\xi \in K$); using Chapter 1, Exercise 15 and Exercise 20, show that if $1 \otimes x = 0$ in $K \otimes M$ then $1 \otimes x = 0$ in $A\xi \otimes M$ for some $\xi \neq 0$. Deduce that $\xi^{-1}x = 0$.]

Proof. Let A be an integral domain. Let M be an A-module and let $T(M) \subseteq M$ be the set of torsion elements of M. For x, $y \in T(M)$, there exist a, $a' \in A$, both nonzero, so that $ax = a'y = 0$. Let $\lambda \in A$. Since A is a domain, $aa' \neq 0$ and

$$(aa')(x+y) = (aa')x + (aa')y = a'(ax) + a(a'y) = a' \cdot 0 + a \cdot 0 = 0$$
$$a(\lambda x) = (a\lambda)x = \lambda(ax) = \lambda \cdot 0 = 0.$$

So, we see that $x + y$ and λx lie in $T(M)$. This shows $T(M)$ is an A-submodule of M.

i) Suppose $\overline{x}$ in $M/T(M)$ is annihilated by some nonzero $a \in A$. Since $a\overline{x} = 0$ in $M/T(M)$, we must have $ax \in T(M)$. Since $ax \in T(M)$, there exists a nonzero $a' \in A$ so that $a'(ax) = 0$. Then $(a'a)x = 0$ and since A is a domain, $a'a$ is nonzero. It follows that $x \in T(M)$ so $\overline{x} = 0$ in $M/T(M)$. This says the torsion submodule of $M/T(M)$ is zero, so $M/T(M)$ is torsion-free.

ii) Suppose $f : M \to N$ is a module homomorphism. Let $x \in T(M)$. Since $x \in T(M)$, there exists a nonzero $a \in A$ so that $ax = 0$. Then $af(x) = f(ax) = f(0) = 0$, so $f(x) \in T(N)$. Since $x \in T(M)$ is arbitrary, $f(T(M)) \subseteq T(N)$.

iii) Suppose $0 \to M' \xrightarrow{f} M \xrightarrow{g} M''$ is an exact sequence of A-modules. By part (ii), this induces maps $f' : T(M') \to T(M)$, $g' : T(M) \to T(M'')$.

We note that
$$\ker(f') = \ker(f) \cap T(M') = \{0\},$$
since f is injective. Thus f' is injective.

We also note that
$$\ker(g') = \ker(g) \cap T(M) = \mathrm{im}(f) \cap T(M) = \mathrm{im}(f').$$

This shows that sequence $0 \to T(M') \xrightarrow{f'} T(M) \xrightarrow{g'} T(M'')$ is exact.

iv) Let M be any A-module. Let K be the field of fractions of A. Consider the map $\varphi : M \to K \otimes_A M$ given by $x \mapsto 1 \otimes_A x$.

If $x \in T(M)$, then there exists $a \in A$, $a \neq 0$, so that $ax = 0$. Then

$$\varphi(x) = 1 \otimes_A x = (1/a)a \otimes_A x = (1/a) \otimes_A ax = (1/a) \otimes_A 0 = 0.$$

So, $x \in \ker\varphi$.

Suppose $x \in \ker\varphi$. The field K is the direct limit of submodules $A\xi$ for $\xi \in K$. (Separate proof for $\mathbb{Q}$.) Hence $1 \otimes x = 0 \in K \otimes M = (\varinjlim_{\xi \in K} A\xi) \otimes M = \varinjlim_{\xi \in K}(A\xi \otimes M)$, the latter equality by Exercise 20 in Chapter 1. By Exercise 15 in Chapter 1, $1 \otimes x = 0$ in $A\xi \otimes M$ for some $\xi \in K$. Then we have $0 = 1 \otimes x = \xi^{-1}\xi \otimes x = \xi \otimes \xi^{-1}x$, which forces $\xi^{-1}x = 0$. So, $x \in T(M)$.

□

Exercise 13. Let S be a multiplicatively closed subset of an integral domain A. In the notation of Exercise 12, show that $T(S^{-1}M) = S^{-1}(TM)$. Deduce that the following are equivalent:

i) M is torsion-free

ii) $M_{\mathfrak{p}}$ is torsion-free for all prime ideals $\mathfrak{p}$.

iii) $M_{\mathfrak{m}}$ is torsion-free for all maximal ideals $\mathfrak{m}$.

Proof. Let S be a multiplicatively closed subset of an integral domain A and let M be an A module. Let $T(M)$ be the torsion submodule of M and let $T(S^{-1}M)$ be the torsion submodule of $S^{-1}M$.

If $0 \in S$, then both $S^{-1}M$ and $S^{-1}(TM)$ are the zero module, so the equality is immediate. So, suppose S doesn't contain zero.

Let $m/s \in T(S^{-1}M)$. Then there exists $a \in A$, $a \neq 0$, so that $a \cdot m/s = am/s = 0$ in $S^{-1}M$. So there exists $t \in S$ so that $tam = 0$. Since S doesn't contain zero, $t \neq 0$, and since A is a domain, $ta \neq 0$. It follows that $m \in TM$ and $m/s \in S^{-1}(TM)$.

On the other hand, let $m/s \in S^{-1}(TM)$. Then $m/s = t/s'$ in $S^{-1}(TM)$ with $t \in TM$ and $s' \in S$. By the definition of $S^{-1}(TM)$, there exists $s'' \in S$ so that $s''(s'm - ts) = 0$, or $s's''m = tss''$. Since $t \in TM$, there exists $a \in A$, $a \neq 0$, so that $at = 0$. Then $as's''m = atss'' = 0$. Then m/s lies in $T(S^{-1}M)$.

The two inclusions show that $T(S^{-1}M) = S^{-1}(TM)$.

i) $\Rightarrow$ ii) Suppose M is torsion free. Then for any multiplicatively closed set $S \subseteq A$, we have

$$T(S^{-1}M) = S^{-1}(TM) = S^{-1}(0) = 0.$$

So $S^{-1}M$ is torsion free. In particular, $M_{\mathfrak{p}}$ is torsion free for all prime ideals $\mathfrak{p}$.

ii) $\Rightarrow$ iii) Since every maximal ideal is a prime ideal, this follows immediately.

iii) $\Rightarrow$ i) Let $\mathfrak{m}$ be any maximal ideal in A. Then $T(M)_{\mathfrak{m}} = T(M_{\mathfrak{m}}) = 0$. By Proposition 3.8, $T(M) = 0$, so M is torsion free.

□

Exercise 14. Let M be an A-module and $\mathfrak{a}$ an ideal of A. Suppose that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \supseteq \mathfrak{a}$. Prove that $M = \mathfrak{a}M$. [Pass to the $A/\mathfrak{a}$-module $M/\mathfrak{a}M$ and use (3.8).]

Proof. Let M be an A-module and $\mathfrak{a}$ an ideal of A. Suppose that $M_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m} \supseteq \mathfrak{a}$.

Consider the quotient module $M/\mathfrak{a}M$. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{a}$. Then $(M/\mathfrak{a}M)_\mathfrak{m} \cong M_\mathfrak{m}/\mathfrak{a}M_\mathfrak{m} = 0$ since $M_\mathfrak{m} = 0$. Since there is a one-to-one correspondence between maximal ideals in $A/\mathfrak{a}$ and maximal ideals in A containing $\mathfrak{a}$, we have that $(M/\mathfrak{a}M)_\mathfrak{m} = 0$ for all maximal ideals in $A/\mathfrak{a}$. By Proposition 3.8, $M/\mathfrak{a}M = 0$. But this means $M = \mathfrak{a}M$. □

Exercise 15. Let A be a ring, and let F be the A-module A^n. Show that every set of n generators of F is a basis of F. [Let $x_1, \ldots, x_n$ be a set of generators and $e_1, \ldots, e_n$ the canonical basis of F. Define $\varphi : F \to F$ by $\varphi(e_i) = x_i$. Then φ is surjective and we have to prove that it is an isomorphism. By (3.9) we may assume that A is a local ring. Let N be the kernel of φ and let $k = A/\mathfrak{m}$ be the residue field of A. Since F is a flat A-module, the exact sequence $0 \to N \to F \xrightarrow{\varphi} F \to 0$ gives an exact sequence $0 \to k \otimes N \to k \otimes F \xrightarrow{1\otimes\varphi} k \otimes F \to 0$. Now $k \otimes F = k^n$ is an n-dimensional vector space over k; $1 \otimes \varphi$ is surjective, hence bijective, hence $k \otimes N = 0$.

Also, N is finitely generated, by Chapter 2, Exercise 12, hence $N = 0$ by Nakayama's lemma. Hence φ is an isomorphism.]

Proof. Let A be a ring and let F be the A-module A^n. Let $\{x_1, \ldots, x_n\}$ be a set of generators of F and let $e_1, \ldots, e_n$ the canonical basis of F.

Mark, finish this one.

□

Exercise 16. Let B be a flat A-algebra. Then the following conditions are equivalent:

i) $\mathfrak{a}^{ec} = \mathfrak{a}$ for all ideals $\mathfrak{a}$ in A.

ii) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

iii) For every maximal ideal $\mathfrak{m}$ of A we have $\mathfrak{m}^e \neq (1)$.

iv) If M is any non-zero A-module, then $M_B \neq 0$.

v) For every A-module M, the mapping $x \mapsto 1 \otimes x$ of M into M_B is injective.

B is said to be *faithfully flat* over A.

Proof. (i)⇒(ii) Suppose $\mathfrak{a}^{ec} = \mathfrak{a}$ for all ideals $\mathfrak{a}$ in A. Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then $\mathfrak{p}^e$ is a prime ideal in B and $\mathfrak{p}^{ec} = \mathfrak{p}$, so $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

(ii)⇒(iii) Suppose $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Let $\mathfrak{m}$ be a maximal ideal in A. Since $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective, there is a prime ideal $\mathfrak{p} \subseteq B$ so that $\mathfrak{p}^c = \mathfrak{m}$. But then $\mathfrak{m}^e \subseteq \mathfrak{p}^{ce} \subseteq \mathfrak{p}$, so $\mathfrak{m}^e \neq (1)$.

(iii)⇒(iv) Suppose for every maximal ideal $\mathfrak{m}$ of A we have $\mathfrak{m}^e \neq (1)$. Let M be any non-zero A-module. Let x be a non-zero element of M and let $M' = Ax$. The sequence of A-modules

$$0 \to M' \to M \to M/M' \to 0$$

is exact, and since B is flat over A, we have

$$0 \to M'_B \to M_B \to (M/M')_B \to 0$$

is exact, too. To show $M_B \neq 0$, it's sufficient to show that $M'_B \neq 0$.

The obvious map $A \to Ax$ is surjective. If $\mathfrak{a}$ is the kernel of this map, then the sequence

$$0 \to \mathfrak{a} \to A \to Ax \to 0$$

is exact, from which is follows that $M' = A/\mathfrak{a}$ for some ideal $\mathfrak{a} \neq (1)$ Hence $M'_B \cong B/\mathfrak{a}^e$. Now $\mathfrak{a} \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, hence $\mathfrak{a}^e \subseteq \mathfrak{m}^e \neq (1)$. Hence $M'_B \neq 0$.

(iv)⇒(v) Suppose that if M is any non-zero A-module, then $M_B \neq 0$.

Let M' be the kernel of $M \to M_B$. Since B is flat over A, the sequence

$$0 \to M'_B \to M_B \to (M_B)_B$$

is exact. But by Chapter 2, Exercise 13, with $N = M_B$, the mapping $M_B \to (M_B)_B$ is injective. So $M'_B = 0$ and therefore $M' = 0$. So, the map $x \mapsto 1 \otimes x$ of M into M_B is injective.

(v)⇒(i) Suppose that for every A-module M, the mapping $x \mapsto 1 \otimes x$ of M into M_B is injective. Let $\mathfrak{a}$ be any ideal in A. If we take M to be the A-module $A/\mathfrak{a}$, then by hypothesis the mapping $A/\mathfrak{a} \to (A/\mathfrak{a})_B = B/\mathfrak{a}^e$ is injective. But this says $\mathfrak{a}^{ec} = \mathfrak{a}$. □

Exercise 17. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms. If $g \circ f$ is flat and g is faithfully flat, then f is flat.

Proof. The map f being flat means that f induces a flat A-algebra structure on B. Let $\varphi : N \to M$ be an injective homomorphism of A-modules. Since C is a flat A-module, $\varphi_C : N_C \to M_C$ is injective.

Consider the diagram

$$\begin{tikzcd} N_B \arrow[r, "\phi_B"] \arrow[d] & M_B \arrow[d] \\ N_C \arrow[r, "\phi_C"] & M_C \end{tikzcd}$$

Note that

$$C \otimes_B N_B = C \otimes_B (B \otimes_A N) = (C \otimes_B B) \otimes_A N \cong N_C.$$

Thus, $N_B \to C \otimes_B N_B = N_C$ defined by $x \mapsto 1 \otimes x$ is the map defined in Exercise 3.16(v). Since g is faithfully flat, this map is injective. Similarly, $M_B \to C \otimes_B M_B = M_C$ is injective.

Since the diagram commutes, ϕ_B is injective, so f is flat.

□

Exercise 18. Let $f : A \to B$ be a flat homomorphism of rings, let $\mathfrak{q}$ be a prime ideal of B and let $\mathfrak{p} = \mathfrak{q}^c$. Then $f^* : \operatorname{Spec}(B_\mathfrak{q}) \to \operatorname{Spec}(A_\mathfrak{p})$ is surjective.

Proof. Let $f : A \to B$ be a flat homomorphism of rings. Let $S = A \setminus \mathfrak{p}$ and $T = B \setminus \mathfrak{q}$. Then $B_\mathfrak{p} = f(S)^{-1}B$, $B_\mathfrak{q} = T^{-1}B$, and since $f(s) \in T$, there is an induced map $A_\mathfrak{p} \to B_\mathfrak{q}$ given by $a/s \mapsto f(a)/f(s)$. Also, Exercise 3 gives an isomorphism

$$B_\mathfrak{q} = T^{-1}B \cong U^{-1}(f(S)^{-1}B) = U^{-1}(B_\mathfrak{p}).$$

where $U = \{1/t \in B_\mathfrak{p} : t \in T\}$. Then we can take a map

$$g : A_\mathfrak{p} \to B_\mathfrak{p} \to U^{-1}(B_\mathfrak{p}) = B_\mathfrak{q} \text{ by } x/s \mapsto f(x)/f(s) \mapsto f(x)/f(s).$$

By Proposition 3.10, $B_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ by $A_\mathfrak{p} \to B_\mathfrak{p}$. By Corollary 3.6, $B_\mathfrak{q}$ is a flat $B_\mathfrak{p}$ module. Note that $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}A_\mathfrak{p}$. Then $g(\mathfrak{p}A_\mathfrak{p}) \subseteq \mathfrak{q}B_\mathfrak{q}$. Thus $(\mathfrak{p}A_\mathfrak{p})^e \neq (1)$. Hence g is faithfully flat by Exercise 16(iii), thus $\operatorname{Spec}(B_\mathfrak{q}) \to \operatorname{Spec}(A_\mathfrak{p})$ is surjective, by Exercise 16(i). □

Exercise 19. Let A be a ring, M an A-module. The *support* of M is defined to be the set $\mathrm{Supp(M)}$ of prime ideals $\mathfrak{p}$ of A such that $M_\mathfrak{p} \neq 0$. Prove the following results:

i) $M \neq 0 \Leftrightarrow \mathrm{Supp(M)} \neq \emptyset$.

ii) $V(\mathfrak{a}) = \mathrm{Supp(A/\mathfrak{a})}$.

iii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then $\mathrm{Supp(M)} = \mathrm{Supp(M')} \cup \mathrm{Supp(M'')}$.

iv) if $M = \sum M_i$, then $\mathrm{Supp(M)} = \bigcup \mathrm{Supp(M_i)}$.

v) If M is finitely generated, then $\mathrm{Supp(M)} = V(\operatorname{Ann}(M))$ (and is therefore a closed subset of $\operatorname{Spec}(A)$).

vi) If M, N are finitely generated, then $\mathrm{Supp(M \otimes_A N)} = \mathrm{Supp(M)} \cap \mathrm{Supp(N)}$.

vii) If M is finitely generated and $\mathfrak{a}$ is an ideal of A, then $\mathrm{Supp(M/\mathfrak{a}M)} = V(\mathfrak{a} + \operatorname{Ann}(M))$.

viii) If $f : A \to B$ is a ring homomorphism and M is a finitely generated A-module, then $\mathrm{Supp(B \otimes_A M)} = {f^*}^{-1}(\mathrm{Supp(M)})$.

Proof. □

Exercise 20. Let $f : A \to B$ be a ring homomorphism, $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the associated mapping. Show that

i) Every prime ideal of A is a contracted ideal $\Leftrightarrow$ f^* is surjective.

ii) Every prime ideal of B is an extended ideal $\Rightarrow$ f^* is injective.

Is the converse of (ii) true?

i) *Proof.* Let $f : A \to B$ be a ring homomorphism, $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the associated mapping.

($\Rightarrow$) Suppose every prime ideal of A is a contracted ideal. Let $\mathfrak{p} \subseteq A$ be a prime ideal. Since every ideal in A is a contracted ideal, there exists a prime ideal $\mathfrak{q} \subseteq B$ so that $\mathfrak{q}^c = \mathfrak{p}$. But then $f^*(\mathfrak{q}) = \mathfrak{p}$. It follows that $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

($\Leftarrow$) Suppose f^* is surjective. Let $\mathfrak{p} \subseteq A$ be a prime ideal. Considering $\mathfrak{p}$ as a point in $\mathrm{Spec}(A)$, there exists a prime ideal $\mathfrak{q} \subseteq B$ so that $f^*(\mathfrak{q}) = \mathfrak{p}$. But this says that $\mathfrak{q}^c = \mathfrak{p}$. So, every prime ideal in A is a contracted ideal. □

ii) *Proof.* Suppose every prime ideal of B is an extended ideal. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be prime ideals in B and suppose that $f^*(\mathfrak{q}) = f^*(\mathfrak{q}')$. Since every prime ideal in B is an extended ideal, there exist prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ so that $\mathfrak{q} = \mathfrak{p}^e$ and $\mathfrak{q}' = \mathfrak{p}'^e$. Then

$$\mathfrak{p}^{ec} = f^*(\mathfrak{p}^e) = f^*(\mathfrak{q}) = f^*(\mathfrak{q}') = f^*(\mathfrak{p}'^e) = \mathfrak{p}'^{ec}.$$

Since $\mathfrak{p}^{ec} = \mathfrak{p}'^{ec}$, we have

$$\mathfrak{q} = \mathfrak{p}^e = \mathfrak{p}^{ece} = \mathfrak{p}'^{ece} = \mathfrak{p}'^e = \mathfrak{q}'.$$

by Proposition 1.17(ii). Hence, f^* is injective. □

Exercise 21. i) Let A be a ring, S a multiplicatively closed subset of A, and $\varphi : A \to S^{-1}A$ the canonical homomorphism. Show that $\varphi^* : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is a homeomorphism of $\mathrm{Spec}(S^{-1}A)$ onto its image in $X = \mathrm{Spec}(A)$. Let this image be denoted by $S^{-1}X$. In particular, if $f \in A$, the image of $\mathrm{Spec}(A_f)$ in X is the basic open set X_f.

ii) Let $f : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$, and let $f^* : Y \to X$ be the mapping associated with f. Identifying $\operatorname{Spec}(S^{-1}A)$ with its canonical image $S^{-1}X$ in X, and $\operatorname{Spec}(S^{-1}B) = \operatorname{Spec}(f(S)^{-1}B)$ with its canonical image $S^{-1}Y$ in Y, show that
$S^{-1}f^* : \operatorname{Spec}(S^{-1}B) \to \operatorname{Spec}(S^{-1}A)$ is the restriction of f^* to $S^{-1}Y$, and that $S^{-1}Y = f^{*-1}(S^{-1}X)$.

iii) Let $\mathfrak{a}$ be an ideal of A and let $\mathfrak{b} = \mathfrak{a}^e$ be its extension in B. Let $\tilde{f} : A/\mathfrak{a} \to B/\mathfrak{b}$ be the homomorphism induced by f. If $\operatorname{Spec}(A/\mathfrak{a})$ is identified with its canonical image $V(\mathfrak{a})$ in X, and $\operatorname{Spec}(B/\mathfrak{b})$ with its image $V(\mathfrak{b})$ in Y, show that $\tilde{f}^*$ is the restriction of f^* to $V(\mathfrak{b})$.

iv) Let $\mathfrak{p}$ be a prime ideal of A. Take $S = A \setminus \mathfrak{p}$ in (ii) and then reduce mod $S^{-1}\mathfrak{p}$ as in (iii). Deduce that the subspace $f^{*-1}(\mathfrak{p})$ of Y is naturally homeomorphic to $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) = \operatorname{Spec}(k(\mathfrak{p}) \otimes_A B))$, where $k(\mathfrak{p})$ is the residue field of the local ring $A_\mathfrak{p}$. $\operatorname{Spec}(k(\mathfrak{p}) \otimes_A B))$ is called the *fiber* of f^* over $\mathfrak{p}$.

i) *Proof.* Let A be a ring, S a multiplicatively closed subset of A, and $\varphi : A \to S^{-1}A$ the canonical homomorphism. Consider the associated map $\varphi^* : \operatorname{Spec}(S^{-1}A) \to \operatorname{Spec}(A)$. By Exercise 21(i) in Chapter 1, φ^* is continuous. Let $S^{-1}X$ denote the image of φ^*, so that $\varphi^* : \operatorname{Spec}(S^{-1}A) \to S^{-1}X$ is continuous and surjective.

Let $V(\mathfrak{q})$ be a closed set in $\operatorname{Spec}(S^{-1}A)$. Since every ideal in $S^{-1}A$ is an extended ideal by Proposition 3.11(i), there exists a prime ideal $\mathfrak{p}$ in A so that $\mathfrak{p}^e = \mathfrak{q}$. By Exercise 21 in Chapter 1,

$$\overline{\phi^*(V(\mathfrak{q}))} = \overline{\phi^*(V(\mathfrak{p}^e))} = V(\mathfrak{p}^{ec})$$

Mark, finish this.

□

ii) *Proof.* Let $f : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$, and let $f^* : Y \to X$ be the mapping associated with f. Identify $\operatorname{Spec}(S^{-1}A)$ with its canonical image $S^{-1}X$ in X, and $\operatorname{Spec}(S^{-1}B) = \operatorname{Spec}(f(S)^{-1}B)$ with its canonical image $S^{-1}Y$ in Y.

The commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ S^{-1}A \arrow[r, "S^{-1}f"] & f(S)^{-1}B \end{tikzcd}$$

gives rise to a commutative diagram

$$\begin{tikzcd}
X = \operatorname{Spec}(A) & & \operatorname{Spec}(B) = Y \arrow[ll, "f^*"'] \\
S^{-1}X = \operatorname{Spec}(S^{-1}A) \arrow[u] & & \operatorname{Spec}(f(S)^{-1}B) = S^{-1}Y \arrow[ll, "S^{-1}f^*"'] \arrow[u]
\end{tikzcd}$$

Then it's clear that $S^{-1}f^*$ is the restriction of f^* to $S^{-1}Y$ and $S^{-1}Y$ maps into $S^{-1}X$ under this restriction. $\square$

iii) *Proof.* Let $\mathfrak{a}$ be an ideal of A and let $\mathfrak{b} = \mathfrak{a}^e$ be its extension in B. Let $\tilde{f} : A/\mathfrak{a} \to B/\mathfrak{b}$ be the homomorphism induced by f. Let $\operatorname{Spec}(A/\mathfrak{a})$ be identified with its canonical image $V(\mathfrak{a})$ in X, and $\operatorname{Spec}(B/\mathfrak{b})$ with its image $V(\mathfrak{b})$ in Y.

The commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
A/\mathfrak{a} \arrow[r, "f"] & B/\mathfrak{b}
\end{tikzcd}$$

gives rise to a commutative diagram

$$\begin{tikzcd}
X = \operatorname{Spec}(A) & \operatorname{Spec}(B) = Y \arrow[l, "f^*"'] \\
V(\mathfrak{a}) = \operatorname{Spec}(A/\mathfrak{a}) \arrow[u] & \operatorname{Spec}(B/\mathfrak{b}) = V(\mathfrak{b}) \arrow[l, "\tilde{f}^*"'] \arrow[u]
\end{tikzcd}$$

Then it's clear that $\tilde{f}^*$ is the restriction of f^* to $V(\mathfrak{b})$ and $V(\mathfrak{b})$ maps into $V(\mathfrak{a})$ under this restriction. $\square$

iv) *Proof.* Let $\mathfrak{p}$ be a prime ideal of A. Take $S = A \setminus \mathfrak{p}$ in (ii) and then reduce mod $S^{-1}\mathfrak{p}$ as in (iii):

The commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
A_{\mathfrak{p}} \arrow[r, "S^{-1}f"] \arrow[d] & B_{\mathfrak{p}} \arrow[d] \\
A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \arrow[r, "S^{-1}\tilde{f}"] & B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}
\end{tikzcd}$$

gives rise to a commutative diagram

$$\begin{tikzcd}
X = \mathrm{Spec}(A) & \mathrm{Spec}(B) = Y \arrow[l, "f^*"'] \\
\mathrm{Spec}(A_{\mathfrak{p}}) \arrow[u] & \mathrm{Spec}(B_{\mathfrak{p}}) \arrow[l, "(S^{-1}f)^*"'] \arrow[u] \\
\mathrm{Spec}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \arrow[u] & \mathrm{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) \arrow[l, "(S^{-1}\tilde{f})^*"'] \arrow[u]
\end{tikzcd}$$

From this commutative diagram, it follows that the subspace ${f^*}^{-1}(\mathfrak{p})$ of Y is naturally homeomorphic to $\mathrm{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) = \mathrm{Spec}(k(\mathfrak{p}) \otimes_A B))$, where $k(\mathfrak{p})$ is the residue field of the local ring $A_{\mathfrak{p}}$. $\mathrm{Spec}(k(\mathfrak{p}) \otimes_A B))$ is called the *fiber* of f^* over $\mathfrak{p}$. □

Mark, check this last paragraph out.

Exercise 22. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Then the canonical image of $\mathrm{Spec}(A_{\mathfrak{p}})$ in $\mathrm{Spec}(A)$ is equal to the intersection of all the open neighborhoods of $\mathfrak{p}$ in $\mathrm{Spec}(A)$.

Proof. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Let $f : A \to A_{\mathfrak{p}}$ be the localization homomorphism and let $f^* : \mathrm{Spec}(A_{\mathfrak{p}}) \to \mathrm{Spec}(A)$ be the associated map of spectra.

Let $\mathfrak{q}$ be any point in $\mathrm{Spec}(A_{\mathfrak{p}})$. By Proposition 3.11(i), $\mathfrak{q}$ is an extended ideal, so there exists a prime ideal $\mathfrak{p}' \subset A$ so that $(\mathfrak{p}')^e = \mathfrak{q}$. Then $\mathfrak{p} \supseteq \mathfrak{q}^c = (\mathfrak{p}')^{ec}$

Mark, finish this.

$\square$

Exercise 23. Let A be a ring, let $X = \operatorname{Spec}(A)$ and let U be a basic open set in X, i.e. $U = X_f$ for some $f \in A$.

i) If $U = X_f$, show that the ring $A(U) = A_f$ depends only on U and not on f.

ii) Let $U' = X_g$ be another basic open set such that $U' \subseteq U$. Show that there is an equation of the form $g^n = uf$ for some integer $n > 0$ and some $u \in A$, and use this to define a homomorphism $\rho : A(U) \to A(U')$ (that is, $A_f \to A_g$) by mapping a/f^m to au^m/g^{mn}. Show that ρ depends only on U and U'. This homomorphism is called the *restriction* homomorphism.

iii) If $U = U'$, the ρ is the identity map.

iv) If $U \supseteq U' \supseteq U''$ are basic open sets in X, show that the diagram

$$\begin{tikzcd} A(U) \arrow[rr] \arrow[dr] & & A(U'') \\ & A(U') \arrow[ur] & \end{tikzcd}$$

(in which the arrows are restriction homomorphisms) is commutative.

v) Let $x(= \mathfrak{p})$ be a point of X. Show that

$$\varinjlim_{U \ni x} A(U) \cong A_{\mathfrak{p}}$$

The assignment of the ring $A(U)$ to each basic open set U of X, and the restriction homomorphisms ρ, satisfying the conditions (iii) and (iv) above, constitutes a *presheaf of rings* on the basis of open set $(X_f)_{f \in A}$. (v) says that the stalk of this presheaf at $x \in X$ is the corresponding local ring $A_{\mathfrak{p}}$.

Proof. $\square$

Exercise 24. Show that the presheaf of Exercise 23 has the following property. Let $(U_i)_{i \in I}$ be a covering of X by basic open sets. For each $i \in I$ let $s_i \in A(U_i)$ be such that, for each pair of indices i, j, the images of s_i and s_j in $A(U_i \cap U_j)$ are equal. then there exists a unique $s \in A(= A(X))$ whose image in $A(U_i)$ is s_i, for all $i \in I$. (This essentially implies that the presheaf is a *sheaf*.)

Proof. □

Exercise 25. Let $f : A \to B$, $g : A \to C$ be ring homomorphisms and let $h : A \to B \otimes_A C$ be defined by $h(x) = f(x) \otimes g(x)$. let X, Y, Z, T be prime spectra of A, B, C, $B \otimes_A C$, respectively. Then $h^*(T) = f^Y \cap g^*(Z)$.

[Let $\mathfrak{p} \in X$, and let $k = k(\mathfrak{p})$ be the residue field at $\mathfrak{p}$. By Exericse 21, the fiber $(h^*)^{-1}(\mathfrak{p})$ is the spectrum of $(B \otimes_A C) \otimes_A k \cong (B \otimes_A k) \otimes_k (C \otimes_A k)$. Hence, $\mathfrak{p} \in h^*(T) \Leftrightarrow (B \otimes_A k) \otimes_k (C \otimes_A k) \neq 0 \Leftrightarrow B \otimes_A k \neq 0$ and $C \otimes_A k \neq 0 \Leftrightarrow \mathfrak{p} \in f^*(Y) \cap g^*(Z)$.

Proof. □

Exercise 26. Let $(B_\alpha, g_{\alpha\beta})$ be a direct system of rings and B the direct limit. For each α, let $f_\alpha : A \to B_\alpha$ be a ring homomorphism such that $g_{\alpha\beta} \circ f_\alpha = f_\beta$ whenever $\alpha \leq \beta$. That is, the B_α form a direct system of A-algebras. The f_α induce $f : A \to B$. Show that

$$f^*(\mathrm{Spec}(B)) = \bigcap_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha)).$$

Proof. □

Exercise 27.

i) Let $f_\alpha : A \to B_\alpha$ be any family of A-algebras and let $f : A \to B$ be their tensor product over A (Chapter 2, Exercise 23). Then

$$f^*(\mathrm{Spec}(B)) = \bigcap_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha)).$$

[Use Exercises 25 and 26.]

ii) Let $f_\alpha : A \to B_\alpha$ be any finite family of A-algebras and let $B = \Pi_\alpha B_\alpha$. Define $f : A \to B$ by $f(x) = (f_\alpha(x))$. Then $f^*(\mathrm{Spec}(B)) = \cup_\alpha f_\alpha^*(\mathrm{Spec}(B_\alpha))$.

iii) Hence the subsets of $X = \mathrm{Spec}(A)$ of the form $f^*(\mathrm{Spec}(B))$, where $f : A \to B$ is a ring homomorphism, satisfy the axioms for closed sets in a topological space. The associated topology is the *constructible* topology on X. It is finer that the Zariski topology (i.e., there are more open sets, or equivalently more closed sets).

iv) Let X_C denote the set X endowed with the constructible topology. Show that X_C is quasi-compact.

Proof. □

Exercise 28. (Continuation of the Exercise 27.)

i) For each $g \in A$, the set X_g is both open and closed in the constructible topology.

ii) Let C' denote the smallest topology on X for which the sets X_g are both open and closed, and let $X_{C'}$ denote the set X endowed with this topology. Show that $X_{C'}$ is Hausdorff.

iii) Deduce that the identity mapping $X_C \to X_{C'}$ is a homeomorphism. Hence a subset E of X is of the form $f^*(\mathrm{Spec}(B))$ for some $f : A \to B$ if and only if it is closed in the topology C'.

iv) The topological space X_C is compact, Hausdorff and totally disconnected.

Proof. $\square$

Exercise 29. Let $f : A \to B$ be a ring homomorphism. Show that $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a continuous *closed* mapping (i.e., maps closed sets to closed sets) for the constructible topology.

Proof. $\square$

Exercise 30. Show that the Zariski topology and the constructible topology on $\mathrm{Spec}(A)$ are the same if and only if $A/\mathfrak{N}$ is absolutely flat (where $\mathfrak{N}$ is the nilradical of A).

Proof. $\square$

Chapter 4

Primary Decomposition

Exercises

Exercise 1. If an ideal $\mathfrak{a}$ has a primary decomposition, then $\operatorname{Spec}(A/\mathfrak{a})$ has only finitely many irreducible components.

Proof. Let A be a ring and let the ideal $\mathfrak{a}$ in A have a primary decomposition $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$.

$$\operatorname{rad}(\mathfrak{a}) = \operatorname{rad}(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n) = \operatorname{rad}(\mathfrak{q}_1) \cap \cdots \cap \operatorname{rad}(\mathfrak{q}_n) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n.$$

Since the radical of $\mathfrak{a}$ is the intersection of all prime ideals containing $\mathfrak{a}$, the prime ideals $\mathfrak{p}_1, \ldots \mathfrak{p}_n$ are the minimal ideals of $\mathfrak{a}$. So, $V(\mathfrak{p}_1), \ldots, V(\mathfrak{p}_n)$ are the irreducible components of $\operatorname{Spec}(A/\mathfrak{a})$. □

Exercise 2. If $\mathfrak{a}$ is decomposable and $\mathfrak{a} = \operatorname{rad}(\mathfrak{a})$, then $\mathfrak{a}$ has no embedded prime ideals.

Proof. Suppose $\mathfrak{a}$ is decomposable and $\mathfrak{a} = \operatorname{rad}(\mathfrak{a})$. Then $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ where $\mathfrak{p}_i$ are the prime ideals belonging to $\mathfrak{a}$. So, this is the primary decomposition of $\mathfrak{a}$. Suppose $\mathfrak{p}_j$ is an embedded prime ideal. Then $\mathfrak{p}_j \supsetneq \mathfrak{p}_k \supset \cap_{i \neq j} \mathfrak{p}_i$. This contradicts the fact that $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ is a minimal primary decomposition of $\mathfrak{a}$. □

Exercise 3. If A is absolutely flat, every primary ideal is maximal.

Proof. Let A be absolutely flat and let $\mathfrak{q}$ be a primary ideal.

Let $\overline{x} \in A/\mathfrak{q}$ is nonzero. Then $x \in A \setminus \mathfrak{q}$. Since A is absolutely flat, every principal ideal is idempotent, so $(x) = (x^2)$. So, there exists $a \in A$ so that $x = ax^2$. Then $x(ax - 1) = 0 \in \mathfrak{q}$, and since $x \notin q$, $(ax-1)^n \in \mathfrak{q}$ for some $n \in \mathbb{N}$. So, $\overline{ax} - \overline{1}$ is nilpotent in $A/\mathfrak{q}$. Then $\overline{ax} = (\overline{ax} - \overline{1}) + \overline{1}$ is a unit, by Exercise 1 in Chapter 1. It follows that $\overline{x}$ is a unit, so $A/\mathfrak{q}$ is a field. Hence $\mathfrak{q}$ is a maximal ideal. So, we have that every primary ideal is maximal. $\square$

Exercise 4. In the polynomial ring $\mathbb{Z}[t]$, the ideal $\mathfrak{m} = (2, t)$ is maximal and the ideal $\mathfrak{q} = (4, t)$ is $\mathfrak{m}$-primary, but is not a power of $\mathfrak{m}$.

Proof. The quotient ring $\mathbb{Z}[t]/(2, t) \cong \mathbb{Z}/2$, a field, so the ideal $\mathfrak{m} = (2, t)$ is maximal. Let $\mathfrak{r}$ be the radical of $\mathfrak{q}$. Since $2^2 = 4 \in \mathfrak{q}$, $2 \in \mathfrak{r}$, hence $(2, t) = \mathfrak{m} \subseteq \mathfrak{r}$. Since $\mathfrak{m}$ is maximal, $\mathfrak{m} = \mathfrak{r}$. Looking at the quotient ring, $\mathbb{Z}[t]/(4, t) \cong \mathbb{Z}/(4)$, every zero divisor here is nilpotent. So, $\mathfrak{q}$ is $\mathfrak{m}$-primary. However, $\mathfrak{m}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{m}$, so $\mathfrak{q}$ is not a power of $\mathfrak{m}$. $\square$

Exercise 5. In the polynomial ring $K[x, y, z]$ where K is a field and x, y, z are independent indeterminates, let $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$, $\mathfrak{m} = (x, y, z)$; $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime, and $\mathfrak{m}$ is maximal. Let $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$. Show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a reduced primary decomposition of $\mathfrak{a}$. Which components are isolated and which are embedded?

Proof. In the polynomial ring $K[x, y, z]$ where K is a field and x, y, z are independent indeterminates, let $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$, $\mathfrak{m} = (x, y, z)$; $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime, and $\mathfrak{m}$ is maximal. Let $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$.

The ideal $\mathfrak{a}$ equals (x^2, xz, xy, yz). The ideal $\mathfrak{m}^2$ equals $(x^2, y^2, z^2, xy, xz, yz)$. It follows that $\mathfrak{a} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Let $t \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Since $t \in \mathfrak{p}_1$, $t = rx + qy$ for r, $q \in K[x, y, z]$. Since $t = rx + qy \in \mathfrak{p}_2$ and $y \notin \mathfrak{p}_2$, $q \in \mathfrak{p}_2$. Thus, $q = ux + vz$. Then $t = rx + uxy + vyz$. Since $t \in \mathfrak{m}^2$, $rx \in \mathfrak{m}^2$, so $r \in \mathfrak{m}$. Thus, $r = \alpha x + \beta y + \gamma z$. So, we find $t = (\alpha x + \beta y + \gamma z)x + uxy + vyz = \alpha x^2 + (u + \beta)xy + \gamma xz + vyz$. That is, $t \in \mathfrak{a}$. So, we see that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals, they are primary. Since $\operatorname{rad}\left(\mathfrak{m}^2\right) = \mathfrak{m}$ is maximal, $\mathfrak{m}^2$ is also primary. So $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a primary decomposition of $\mathfrak{a}$. Since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are contained in $\mathfrak{m}$, we see that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are isolated components while $\mathfrak{m}^2$ is an embedded component. $\square$

Exercise 6. Let X be an infinite compact Hausdorff space, $C(X)$ the ring of real-valued continuous functions on X (Chapter 1, Exercise 26). Is the zero ideal decomposable in this ring?

Proof. □

Exercise 7. Let A be a ring and let $A[x]$ denote the ring of polynomials in one indeterminate over A. For each ideal $\mathfrak{a}$ of A, let $\mathfrak{a}[x]$ denote the set of all polynomials in $A[x]$ with coefficients in $\mathfrak{a}$.

i) $\mathfrak{a}[x]$ is the extension of $\mathfrak{a}$ to $A[x]$.

ii) If $\mathfrak{p}$ is a prime ideal in A, then $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.

iii) If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in A, then $\mathfrak{q}[x]$ is a $\mathfrak{p}[x]$-primary ideal in $A[x]$.

iv) If $\mathfrak{a} = \cap_{i=1}^{n}\mathfrak{q}_i$ is a minimal primary decomposition in A, then $\mathfrak{a}[x] = \cap_{i=1}^{n}\mathfrak{q}_i[x]$ is a minimal primary decomposition in $A[x]$.

v) If $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$, then $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$.

Proof. Let A be a ring and let $A[x]$ denote the ring of polynomials in one indeterminate over A. For each ideal $\mathfrak{a}$ of A, let $\mathfrak{a}[x]$ denote the set of all polynomials in $A[x]$ with coefficients in $\mathfrak{a}$.

i) Certainly $\mathfrak{a} \subset \mathfrak{a}[x]$, so $\mathfrak{a}^e \subset \mathfrak{a}[x]$. On the other hand, for any $a \in \mathfrak{a}$, $ax^n \in \mathfrak{a}^e$, and it follows that $\mathfrak{a}[x] \subset \mathfrak{a}^e$. Thus, we have $\mathfrak{a}[x]$ is the extension of $\mathfrak{a}$ to $A[x]$.

ii) Let $\mathfrak{p}$ be a prime ideal in A and consider $\mathfrak{p}[x] \subset A[x]$. Then $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$ is a polynomial ring over an integral domain, so this is an integral domain. Thus, $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.

iii) Suppose $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal and consider $\mathfrak{q}[x]$. We must show that every divisor of zero in $A[x]/\mathfrak{q}[x] \cong (A/\mathfrak{q})[x]$ is nilpotent. Let $f = a_0 + a_1x + \cdots + a_nx^n \in (A/\mathfrak{q})[x]$ be a zero divisor. By Exercise 2(iii) in Chapter 1, there exists $a \neq 0$ in $A/\mathfrak{q}$ so that $af = 0$.

    Fix i, $1 \leq i \leq n$. Since $af = 0$, $aa_i = 0$ in $A/\mathfrak{q}$. This implies that $aa_i \in \mathfrak{q}$ and since $\mathfrak{q}$ is a primary ideal, either $a \in \mathfrak{q}$, $a_i \in \mathfrak{q}$ or both a, $a_i \in \operatorname{rad}(\mathfrak{q})$. By construction, $a \neq 0$ in $A/\mathfrak{q}$, so $a \notin \mathfrak{q}$. It follows that $a_i^j \in \mathfrak{q}$ for some $j \in \mathbb{N}$. So, a_i is nilpotent in $A/\mathfrak{q}$. Since $1 \leq i \leq n$ is arbitrary, $a_0, \ldots, a_n$ are nilpotent in $A/\mathfrak{q}$. By Exercise 2(ii) in Chapter 1, f is nilpotent. Since f is an arbitrary divisor of zero in $A[x]/\mathfrak{q}[x]$, every divisor of zero in $A[x]/\mathfrak{q}[x]$ is nilpotent. Hence, $\mathfrak{q}[x]$ is a primary ideal.

iv) Let $\mathfrak{a} = \cap_{i=1}^n \mathfrak{q}_i$ is a minimal primary decomposition in A. Then $\mathfrak{a}[x] = (\cap_{i=1}^n \mathfrak{q}_i)[x] = \cap_{i=1}^n \mathfrak{q}_i[x]$ and by (iii), each $\mathfrak{q}_i[x]$ is a primary ideal. Since $\cap_{i=1}^n \mathfrak{q}_i$ is a minimal primary decomposition, all the $\mathfrak{p}_i = \operatorname{rad}(\mathfrak{q}_i)$ are distinct, so $\operatorname{rad}(\mathfrak{q}_i[x]) = \mathfrak{p}_i[x]$ are all distinct.

v)

□

Exercise 8. Let k be a field. Show that in the polynomial ring $k[x_1, \ldots, x_n]$ the ideals $\mathfrak{p}_i = (x_1, \ldots, x_i)$, $1 \leq i \leq n$, are prime and all their powers are primary. [Hint: Use Exercise 7.]

Proof. Let k be a field. Consider the polynomial ring $k[x_1, \ldots, x_n]$ and the ideals $\mathfrak{p}_i = (x_1, \ldots, x_i)$, $1 \leq i \leq n$.

Since $k[x_1, \ldots, x_n]/\mathfrak{p}_i \cong k[x_{i+1}, \ldots, x_n]$ (or k if $i = n$), which are all integral domains, the ideals $\mathfrak{p}_i = (x_1, \ldots, x_i)$, $1 \leq i \leq n$, are prime ideals.

We proceed with the remainder of the proof by induction on n. For $n = 1$, we have $\mathfrak{p}_1 = (x_1)$ in $k[x_1]$. The quotient ring $k[x_1]/\mathfrak{p}_1 \cong k$, so $\mathfrak{p}_1$ is a maximal ideal. Thus, $\mathfrak{p}_1^m$ is primary for all m by Proposition 4.2.

Assume the result is true for $n-1$ and consider the prime ideals $\mathfrak{p}_i = (x_1, \ldots, x_i)$ in $k[x_1, \ldots, x_n]$. By hypothesis, for $1 \leq i \leq n-1$, all powers of $\mathfrak{p}'_i = (x_1, \ldots, x_i)$ are primary in $k[x_1, \ldots, x_{n-1}]$. By Exercise 7, all powers of $\mathfrak{p}'_i[x_n] \cong \mathfrak{p}_i$ are primary in $k[x_1, \ldots, x_{n-1}][x_n] \cong k[x_1, \ldots, x_n]$. For $i = n$, $\mathfrak{p}_n = (x_1, \ldots, x_n)$ is a maximal ideal, so all its powers are primary by Proposition 4.2. □

Exercise 9. In a ring A, let $D(A)$ denote the set of prime ideals $\mathfrak{p}$ which satisfy the following condition: there exists $a \in A$ such that $\mathfrak{p}$ is minimal in the set of prime ideals containing $(0 : a)$. Show that $x \in A$ is a zero divisor $\Leftrightarrow x \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$.

Let S be a multiplicatively closed subset of A, and identify $\operatorname{Spec}(S^{-1}A)$ with its image in $\operatorname{Spec}(A)$ (Chapter 3, Exercise 21). Show that

$$D(S^{-1}A) = D(A) \cap \operatorname{Spec}(S^{-1}A).$$

If the zero ideal has a primary decomposition, show that $D(A)$ is the set of associated prime ideals of 0.

Proof. Let A be a ring and let $D(A)$ denote the set of prime ideals $\mathfrak{p}$ which satisfy the following condition: there exists $a \in A$ such that $\mathfrak{p}$ is minimal in the set of prime ideals containing $(0 : a)$.

□

Exercise 10. For any prime ideal $\mathfrak{p}$ in a ring A, let $S_{\mathfrak{p}}(0)$ denote the kernel of the homomorphism $A \to A_{\mathfrak{p}}$. Prove that

i) $S_{\mathfrak{p}}(0) \subseteq \mathfrak{p}$

ii) $r(S_{\mathfrak{p}}(0)) = \mathfrak{p} \Leftrightarrow \mathfrak{p}$ is a minimal prime ideal of A.

iii) If $\mathfrak{p} \supseteq \mathfrak{p}'$, then $S_{\mathfrak{p}}(0) \subseteq S_{\mathfrak{p}'}(0)$.

iv) $\bigcap_{\mathfrak{p} \in D(A)} S_{\mathfrak{p}}(0) = 0$, where $D(A)$ is defined in Exercise 9.

Proof. For any prime ideal $\mathfrak{p}$ in a ring A, let $S_{\mathfrak{p}}(0)$ denote the kernel of the homomorphism $A \to A_{\mathfrak{p}}$.

i) Let $x \in S_{\mathfrak{p}}(0)$. Then $x/1$ lies in the kernel of the homomorphism $A \to A_{\mathfrak{p}}$. This implies there exists $y \notin \mathfrak{p}$ so that $xy = 0$. Since $\mathfrak{p}$ is a prime ideal and $y \notin \mathfrak{p}$, we have $x \in \mathfrak{p}$. This shows $S_{\mathfrak{p}}(0) \subseteq \mathfrak{p}$.

ii) ($\Rightarrow$) Suppose $\operatorname{rad}(S_{\mathfrak{p}}(0)) = \mathfrak{p}$. Suppose $\mathfrak{p}' \subseteq \mathfrak{p}$ is another prime ideal.

   Mark, finish this.

   ($\Leftarrow$) Suppose $\mathfrak{p}$ is a minimal prime ideal of A. By part (i), $S_{\mathfrak{p}}(0) \subseteq \mathfrak{p}$, so that $\operatorname{rad}(S_{\mathfrak{p}}(0)) \subseteq \operatorname{rad}(\mathfrak{p}) = \mathfrak{p}$. Since $\mathfrak{p}$ is a minimal prime ideal, we must have $\operatorname{rad}(S_{\mathfrak{p}}(0)) = \mathfrak{p}$.

iii) Suppose $\mathfrak{p} \supseteq \mathfrak{p}'$. Let $x \in S_{\mathfrak{p}}(0)$. Then in the map $\varphi_{\mathfrak{p}} : A \to A_{\mathfrak{p}}$, $\varphi_{\mathfrak{p}}(x) = 0$. That is $x/1 = 0/1$ in $A_{\mathfrak{p}}$, so there exists $y \in A \backslash \mathfrak{p}$ so that $xy = 0$. But then the same y lies outside $\mathfrak{p}'$, so that $x/1 = 0/1$ in $A_{\mathfrak{p}'}$. That is, $x \in S_{\mathfrak{p}'}(0)$. Since $x \in S_{\mathfrak{p}}(0)$ is arbitrary, $S_{\mathfrak{p}}(0) \subseteq S_{\mathfrak{p}'}(0)$.

iv) Let $D(A)$ denote the set of prime ideals $\mathfrak{p}$ which satisfy the following condition: there exists $a \in A$ such that $\mathfrak{p}$ is minimal in the set of prime ideals containing $(0 : \mathfrak{a})$.

   To show: $\bigcap_{\mathfrak{p} \in D(A)} S_{\mathfrak{p}}(0) = 0$

□

Exercise 11. If $\mathfrak{p}$ is a minimal prime ideal of a ring A, show that $S_{\mathfrak{p}}(0)$ is the smallest $\mathfrak{p}$-primary ideal.

Let $\mathfrak{a}$ be the intersection of the ideals $S_{\mathfrak{p}}(0)$ as $\mathfrak{p}$ runs through the minimal prime ideals of A. Show that $\mathfrak{a}$ is contained in the nilradical of A.

Suppose that the zero ideal is decomposable. Prove that $\mathfrak{a} = 0$ if and only if every prime ideal of 0 is isolated.

Proof. Let $\mathfrak{p}$ be a minimal prime ideal in a ring A and let $S_{\mathfrak{p}}(0)$ denote the kernel of the homomorphism $A \to A_{\mathfrak{p}}$. □

Exercise 12. Let A be a ring, S a multiplicatively closed subset of A. For any ideal $\mathfrak{a}$, let $S(\mathfrak{a})$ denote the contraction of $S^{-1}(\mathfrak{a})$ in A. The ideal $S(\mathfrak{a})$ is called the *saturation* of $\mathfrak{a}$ with respect to S. Prove that

i) $S(\mathfrak{a}) \cap S(\mathfrak{b}) = S(\mathfrak{a} \cap \mathfrak{b})$

ii) $S(r(\mathfrak{a})) = r(S(\mathfrak{a}))$

iii) $S(\mathfrak{a}) = (1) \Leftrightarrow \mathfrak{a}$ meets S.

iv) $S_1(S_2(\mathfrak{a})) = (S_1S_2)(\mathfrak{a})$

If $\mathfrak{a}$ has a primary decomposition, prove that the set of ideals $S(\mathfrak{a})$ (where S runs through all multiplicatively closed subsets of A) is finite.

Proof. Let A be a ring, S a multiplicatively closed subset of A. For any ideal $\mathfrak{a}$, let $S(\mathfrak{a})$ denote the contraction of $S^{-1}(\mathfrak{a})$ in A. The ideal $S(\mathfrak{a})$ is called the *saturation* of $\mathfrak{a}$ with respect to S.

i) For ideals $\mathfrak{a}$, $\mathfrak{b}$ in A, by Proposition 3.4, we have $S^{-1}(\mathfrak{a} \cap \mathfrak{b}) = S^{-1}(\mathfrak{a}) \cap S^{-1}(\mathfrak{b})$. By Exercise 1.18, we have

$$\begin{aligned} S(\mathfrak{a} \cap \mathfrak{b}) &= [S^{-1}(\mathfrak{a} \cap \mathfrak{b})]^c \\ &= [S^{-1}(\mathfrak{a}) \cap S^{-1}(\mathfrak{b})]^c \\ &= [S^{-1}(\mathfrak{a})]^c \cap [S^{-1}(\mathfrak{b})]^c \\ &= S(\mathfrak{a}) \cap S(\mathfrak{b}). \end{aligned}$$

ii)

iii)

iv)

□

Exercise 13. Let A be a ring and $\mathfrak{p}$ be a prime ideal of A. The *nth symbolic power of* $\mathfrak{p}$ is defined to be the ideal (in the notation of Exercise 12)

$$\mathfrak{p}^{(n)} = S_{\mathfrak{p}}(\mathfrak{p}^n)$$

where $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$. Show that

i) $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal;

ii) if $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$-primary component;

iii) if $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ has a primary decomposition, then $\mathfrak{p}^{(m+n)}$ is its $\mathfrak{p}$-primary component;

iv) $\mathfrak{p}^{(n)} = \mathfrak{p}^n \Leftrightarrow \mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary.

Proof. □

Exercise 14. Let $\mathfrak{a}$ be a decomposable ideal in a ring A and let $\mathfrak{p}$ be a maximal element of the set of ideals $(a : x)$, where $x \in A$ and $a \notin \mathfrak{a}$. Show that $\mathfrak{p}$ is a prime ideal belonging to $\mathfrak{a}$.

Proof. □

Exercise 15. Let $\mathfrak{a}$ be a decomposable ideal in a ring A, let Σ be an isolated set o prime ideals belonging to $\mathfrak{a}$, and let $\mathfrak{q}_\Sigma$ be the intersection of the corresponding primary components. Let f be an element of A such that, for each prime ideal $\mathfrak{p}$ belonging to $\mathfrak{a}$, we have $f \in \mathfrak{p} \Rightarrow \mathfrak{p} \notin \Sigma$, and let S_f be the set of all powers of f. Shwo that $\mathfrak{q}_\Sigma = S_f(\mathfrak{a}) = (\mathfrak{a} f^n)$ for all large n.

Proof. □

Exercise 16. If A is a ring in which every ideal has a primary decomposition, show that every ring of fractions $S^{-1}A$ has the same property.

Proof. Let $\mathfrak{a}$ be an ideal in $S^{-1}A$. By Proposition 3.11, every ideal in $S^{-1}A$ is and extended ideal, so there exists and ideal $\mathfrak{b} \subset A$ so that $S^{-1}\mathfrak{b} = \mathfrak{a}$. By hypothesis, $\mathfrak{b}$ has a primary decomposition, so there exist primary ideals $\mathfrak{q}_1, \dots,$ $\mathfrak{q}_m$ in A so that

$$\mathfrak{b} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m.$$

Then

$$\mathfrak{a} = S^{-1}\mathfrak{b} = S^{-1}(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m) = S^{-1}(\mathfrak{q}_1) \cap \cdots \cap S^{-1}(\mathfrak{q}_m).$$

By Proposition 4.8, each ideal $S^{-1}(\mathfrak{q}_i)$ is either a primary ideal in $S^{-1}A$ or is the entire ring $S^{-1}A$. Thus, this gives a primary decomposition of $\mathfrak{a}$. So, any ideal in $S^{-1}A$ has a primary decomposition. □

Exercise 17. Let A be a ring with the following property:
(L1) For every ideal $\mathfrak{a} \neq (1)$ in A and every prime ideal $\mathfrak{p}$, there exists $x \notin \mathfrak{p}$ such that $S_\mathfrak{p}(a) = (a : x)$, where $S_\mathfrak{p} = A \setminus \mathfrak{p}$.

Then every ideal in A is an intersection of (possibly infinitely many) prime ideals.

[Let $\mathfrak{a}$ be an ideal $\neq (1)$ in A, and let $\mathfrak{p}_1$ be a minimal element of the set of prime ideals containing $\mathfrak{a}$. Then $\mathfrak{q}_1 = S_{\mathfrak{p}_1}(\mathfrak{a})$ is $\mathfrak{p}_1$-primary (by Exercise 11), and $\mathfrak{q}_1 = (\mathfrak{a} : x)$ for some $x \notin \mathfrak{p}_1$. Show that $\mathfrak{a} = \mathfrak{q}_1 \cap (\mathfrak{a} + (x))$.

Now let $\mathfrak{a}_1$ be a maximal element of the set of ideals $\mathfrak{b} \supseteq \mathfrak{a}$ such that $\mathfrak{q}_1 \cap \mathfrak{b} = \mathfrak{a}$, and choose $\mathfrak{a}_1$ so that $x \in \mathfrak{a}_1$, and therefore $\mathfrak{a}_1 \nsubseteq \mathfrak{p}_1$. Repeat the construction starting with $\mathfrak{a}_1$, and so on. At the nth stage we have $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{a}_n$ where the $\mathfrak{q}_i$ are primary ideals, $\mathfrak{a}_n$ is maximal among the ideals $\mathfrak{b}$ containing $\mathfrak{a}_{n-1} = \mathfrak{a}_n \cap \mathfrak{q}_n$ such that $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{b}$, and $\mathfrak{a}_n \nsubseteq \mathfrak{p}_n$. If at any stage we have $\mathfrak{a}_n = (1)$, the process stops, and $\mathfrak{a}$ is a finite intersection of primary ideals. If not, continue by transfinite induction, observing that each $\mathfrak{a}_n$ strictly contains $\mathfrak{a}_{n-1}$.]

Proof. □

Exercise 18. Consider the following condition on a ring A:

(L2) Given an ideal $\mathfrak{a}$ and a descending chain $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n \supseteq \cdots$ of multiplicatively closed subsets of A, there exists an integer n such that $S_n(\mathfrak{a}) = S_{n+1}(\mathfrak{a}) = \cdots$. Prove that the following are equivalent:

i) Every ideal in A has a primary decomposition;

ii) A satisfies (L1) and (L2).

[For (i)$\Rightarrow$(ii), use Exercises 12 and 15. For (ii)$\Rightarrow$(i) show, with the notation of the proof of Exercise 17, that if $S_n = S_{\mathfrak{p}_1} \cap \cdots \cap S_{\mathfrak{p}_n}$ then S_n meets $\mathfrak{a}_n$, hence $S_n(\mathfrak{a}_n) = (1)$, and therefore $S_n(\mathfrak{a}) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$. Now use (L2) to show that the construction must terminate after a finite number of steps.]

Proof. □

Exercise 19. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Show that every $\mathfrak{p}$-primary ideal contains $S_{\mathfrak{p}}(0)$, the kernel of the canonical homommorphism $A \to A_{\mathfrak{p}}$.

Suppose that A satisfies the following condition: for every prime ideal $\mathfrak{p}$, the intersection of all $\mathfrak{p}$-primary of A is equal to $S_{\mathfrak{p}}(0)$. (Noetherian rings satisfy this condition: See Chapter 10.) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct prime ideals, none of which is a minimal prime ideal of A. Then there exists an ideal $\mathfrak{a}$ in A whose associated prime ideals are $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.

Proof. □

Exercise 20. Let M be a fixed A-module, N a submodule of M. The *radical of N in M* is defined to be

$$\mathrm{rad}_M(N) = \{x \in A : x^q M \subseteq N \text{ for some } q > 0\}.$$

Show that $\mathrm{rad}_M(N) = \mathrm{rad}(N : M) = \mathrm{rad}\,(\mathrm{Ann}\,(M/N))$ In particular, $r_M(N)$ is an *ideal*.

State and prove the formulas for r_M analogous to (1.13).

Proof. □

Exercise 21. An element $x \in A$ defines an endomorphism ϕ_x of M, namely $m \mapsto xm$. That element x is said to be a *zero-divisor* (resp. *nilpotent*) in M if ϕ_x is not injective (resp. is nilpotent). A submodule Q of M is *primary in* M if $Q \neq M$ and every zero-divisor in M/Q is nilpotent.

Show that if Q is primary in M, then $(Q : M)$ is a primary ideal and hence $r_M(Q)$ is a prime ideal $\mathfrak{p}$. We say that Q is $\mathfrak{p}$-*primary* (in M).

Proof. □

Exercise 22. A *primary decomposition* of N in M is a representation of N as an intersection

$$N = Q_1 \cap \cdots \cap Q_n$$

of primary submodules of M; it is a *minimal primary decomposition* if the ideals $\mathfrak{p}_i = r_M(Q_i)$ are all distinct and if none of the components Q_i can be omitted from the intersection, that is if $Q_i \not\supseteq \bigcap_{j \neq i} Q_j$ $(1 \leq i \leq n)$.

Prove the analogue of (4.5), that the prime ideal $\mathfrak{p}_i$ depend only on N (and M). They are called the *prime ideals belonging to* N *in* M. Show that they are also the prime ideals belonging to 0 in M/N.

Proof. □

Exercise 23. State and prove the analoges of (4.6)–(4.11) inclusive. (There is no loss of generality in taking $N = 0$.)

Proof. □

Chapter 5

Integral Dependence and Valuations

Exercises

Exercise 1. Let $f : A \to B$ be an integral homomorphism of rings. Show that $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a *closed* mapping, i.e. that it maps closed sets to closed sets. (This is a geometrical equivalent of Theorem 5.10.)

Proof. Let $f : A \to B$ be an integral homomorphism of rings and consider the associated morphism of schemes $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

Let $V(\mathfrak{a})$ be any closed set in $\mathrm{Spec}(B)$, where $\mathfrak{a} \subset B$ is an ideal. We show that $f^*(V(\mathfrak{a})) = V(f^{-1}(\mathfrak{a}))$. This will show that f^* is a closed mapping.

First, if $\mathfrak{p} \in V(\mathfrak{a})$ so that $\mathfrak{p} \supseteq \mathfrak{a}$, then $f^{-1}(\mathfrak{p})$ is a prime ideal containing $f^{-1}(\mathfrak{a})$. This shows $f^*(V(\mathfrak{a})) \subseteq V(f^{-1}(\mathfrak{a}))$. In the other direction, let $\mathfrak{p} \in V(f^{-1}(\mathfrak{a}))$, so that $\mathfrak{p}$ is a prime ideal in A containing $f^{-1}(\mathfrak{a})$. Since B is integral over A, by Theorem 5.10, there exists a prime ideal $\mathfrak{q} \subseteq B$ so that $\mathfrak{q} \cap A = f^{-1}(\mathfrak{q}) = \mathfrak{p}$. This shows that $V(f^{-1}(\mathfrak{a})) \subseteq f^*(V(\mathfrak{a}))$. The two inclusions show that $f^*(V(\mathfrak{a})) = V(f^{-1}(\mathfrak{a}))$, so the map f^* is a closed map. □

Exercise 2. Let A be a subring of a ring B such that B is integral over A, and let $f : A \to \Omega$ be a homomorphism A into an algebraically closed field Ω. Show that f be be extended to a homomorphism of B into Ω.

Proof. Mark, finish this using Zorn's Lemma.

□

Exercise 3. Let $f : B \to B'$ be a homomorphism of A-algebras, and let C be an A-algebra. If f is integral, prove that $f \otimes 1 : B \otimes_A C \to B' \otimes_A C$ is integral. (This includes Proposition 5.6 (ii) as a special case.)

Proof. Let $f : B \to B'$ be an integral homomorphism of A-algebras, and let C be an A-algebra. We will identify B with its image in B', so we consider $B \subset B'$.

Consider the homomorphism $f \otimes 1 : B \otimes_A C \to B' \otimes_A C$. It suffices to show that $b' \otimes c \in B' \otimes_A C$ is integral over the image of f. Since f is integral, there is a monic polynomial $p(x) \in B[x]$ so that $p(b') = 0$. Say

$$p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0.$$

Then we have

$$p \otimes 1(b' \otimes c) := p(b') \otimes c = 0 \otimes c = 0.$$

Hence, $b' \otimes c$ is integral over $B \otimes C$. It follows that the map $f \otimes 1 : B \otimes_A C \to B' \otimes_A C$ is integral. □

Exercise 4. Let A be a subring of a ring B such that B is integral over A. Let $\mathfrak{n}$ be a maximal ideal of B and let $\mathfrak{n} = \mathfrak{m} \cap A$ be the corresponding maximal ideal of A. Is $B_{\mathfrak{n}}$ necessarily integral over $A_{\mathfrak{m}}$?

[Consider the subring $k[x^2 - 1]$ of $k[x]$, where k is a field, and let $\mathfrak{n} = (x-1)$. Can the element $1/(x+1)$ be integral?]

Proof. □

Exercise 5. Let $A \subseteq B$ be rings, B integral over A.

i) If $x \in A$ is a unit in B then it is a unit in A.

ii) The Jacobson radical of A is the contraction of the Jacobson radical of B.

Proof. Let $A \subseteq B$ be rings, B integral over A.

i) Let $x \in A$ be a unit in B. Since $x^{-1} \in B$ and B is integral over A, there is a monic polynomial so that

$$(x^{-1})^n + a_{n-1}(x^{-1})^{n-1} + \cdots + a_1x^{-1} + a_0 = 0,$$

with $a_i \in A$ for all i. Then

$$(x^{-1})^n = -\left(a_{n-1}(x^{-1})^{n-1} + a_{n-2}(x^{-1})^{n-2} + \cdots + a_2(x^{-1})^2 + a_1x^{-1} + a_0\right),$$

and

$$x^{-1} = -\left(a_{n-1} + a_{n-2}x + \cdots + a_2x^{n-3} + a_1x^{n-2} + a_0x^{n-1}\right) \in A.$$

So, x is a unit in A.

ii) Let x be in the Jacobson radical of B. By Proposition 1.9, for all $y \in B$, $1 - xy$ is a unit in B. Suppose in addition that $x \in A$. Then $1 - xy$ is a unit in B for all $y \in A \subseteq B$. By part (i), $1 - xy$ is a unit in A for all $y \in A$. Since this is true for all $y \in A$, x lies in the Jacobson radical of A, by Proposition 1.9 again. This shows the contraction of the Jacobson radical of B is the Jacobson radical of A.

□

Exercise 6. Let $B_1, \ldots, B_n$ be integral A-algebras. Show that $\Pi_{i=1}^n B_i$ is an integral A-algebra.

Solution. *Proof.* □

Exercise 7. Let A be a subring of a ring B, such that the set $B \setminus A$ is closed under multiplication. Show that A is integrally closed in B.

Solution. *Proof.* Let A be a subring of a ring B, such that the set $B \setminus A$ is closed under multiplication. Since $B \setminus A$ is multiplicatively closed, if $xy \in A$, then $x \in A$ or $y \in A$.

Let $b \in B$ be integral over A. Say b satisfies the monic polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in A[x]$. We proceed by induction on the degree of this polynomial. For $n = 1$, we have $b + a_0 = 0$, so that $b = -a_0 \in A$.

Assume the result is true for $n < N$.

Suppose that if $b \in B$ is integral and satisfies a monic polynomial in $A[x]$ of degree N. Say $b \in B$ satisfies a polynomial $x^N + a_{N-1}x^{N-1} + \cdots + a_1 x + a_0$. Then $b^N + a_{N-1}b^{N-1} + \cdots + a_1 b + a_0 = 0$, so that

$$b(b^{N-1} + a_{N-1}b^{N-2} + \cdots + a_2 b + a_1) = -a_0 \in A.$$

Since $B \setminus A$ is multiplicatively closed, we either have $b \in A$ and we're finished or $b^{N-1} + a_{N-1}b^{N-1} + \cdots + a_2 b + a_1 \in A$. In this case b satisfies a monic polynomial in $A[x]$ of degree $N - 1$. By induction, $b \in A$.

Thus, we see that A is integrally closed in B. □

Mark, start here.

Exercise 8. a) Let A be a subring of an integral domain B, and let C be the integral closure of A in B. Let f, g be monic polynomials in $B[x]$ such that $fg \in C[x]$. Then $f, g \in C[x]$.

b) Prove the same result without assuming that B (or A) is an integral domain.

Solution. a) *Proof.* Let A be a subring of an integral domain B, and let C be the integral closure of A in B. Let f, g be monic polynomials in $B[x]$ such that $fg \in C[x]$. Since fg is also monic, all its roots are integral over C, and since C is the integral closure of A, all its roots are integral over A. Since these roots are precisely the roots of f and the roots of g, the coefficients of f and g are polynomials in the roots, and are therefore also integral over A. So all the coefficients of f and g lie in C and f, g lie in $C[x]$.

□

b) *Proof.* □

Exercise 9. Let A be a subring of a ring B and let C be the integral closure of A in B. Prove that $C[x]$ is the integral closure of $A[x]$ in $B[x]$. [If $f \in B[x]$ is integral over $A[x]$, then

$$f^m + g_1 f^{m-1} + \cdots + g_m = 0 \quad (g_i \in A[x]).$$

Let r be an integer larger than m and the degrees of $g_1, \ldots, g_m$, and let $f_1 = f - x^r$, so that

$$(f_1 + x^r)^m + g_1(f_1 + x^r)^{m-1} + \cdots + g_m = 0$$

or say

$$f_1^m + h_1 f_1^{m-1} + \cdots + h_m = 0,$$

where $h_m = (x^r)^m + g_1(x^r)^{m-1} + \cdots + g_m \in A[x]$. Now apply Exercise 8 to the polynomials $-f_1$ and $f_1^{m-1} + h_1 f_1^{m-2} + \cdots + h_{m-1}$.]

Solution. *Proof.* □

Exercise 10. (a) A ring homomorphism $f : A \to B$ is said to have the *going-up property* if the conclusion of the going-up theorem Theorem 5.11 holds for B and its subring $f(A)$.

Let $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the mapping associated with f.

Consider the following three statements:

(a) f^* is a closed mapping.

(b) f has the going-up property.

(c) Let $\mathfrak{q}$ be any prime ideal of B and let $\mathfrak{p} = \mathfrak{q}^c$. Then $f^* : \mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(A/\mathfrak{p})$ is surjective.

Prove that (a)⇒(b) $\iff$ (c). (See also Chapter 6, Exercise 11.)

Solution. *Proof.* □

Exercise 10. (b) A ring homomorphism $f : A \to B$ is said to have the *going-down property* if the conclusion of the going-down theorem Theorem 5.16 holds for B and its subring $f(A)$.

Let $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be the mapping associated with f.

Consider the following three statements:

(a) f^* is a open mapping.

(b) f has the going-down property.

(c) For any prime ideal $\mathfrak{q}$ of B, if $\mathfrak{p} = \mathfrak{q}^c$, then $f^* : \operatorname{Spec}(B_\mathfrak{q}) \to \operatorname{Spec}(A_\mathfrak{p})$ is surjective.

Prove the (a)⇒(b) $\iff$ (c). (See also Chapter 7, Exercise 23.)

[To prove that (a)⇒(c), observe that $B_\mathfrak{q}$ is the direct limit of the rings B_t where $t \in B \setminus \mathfrak{q}$; hence, by Chapter 3, Exercise 26, we have $f^*(\operatorname{Spec}(B_\mathfrak{q})) = \cap_t f^*(\operatorname{Spec}(B_t)) = \cap_t f^*(Y_t)$. Since Y_t is an open neighborhood of $\mathfrak{q}$ in Y, and since f^* is open, it follow that $f^*(Y_t)$ is an open neighborhood of $\mathfrak{p}$ in X and therefore contains $\operatorname{Spec}(A_\mathfrak{p})$.

Solution. *Proof.* □

Exercise 11. Let $f : A \to B$ be a flat homomorphism of rings. Then f has the going-down property. [Chapter 3, Exercise 18.]

Solution. *Proof.* □

Exercise 12. Let G be a finite group of automorphisms of a ring A, and let A^G denote the subring of G-invariants, that is of all $x \in A$ such that $\sigma(x) = x$ for all $\sigma \in G$. Prove that A is integral over A^G.

Let S be a multiplicatively closed subset of A such that $\sigma(S) \subseteq S$ for all $\sigma \in G$, and let $S^G = S \cap A^G$. Show that the action of G on A extends to an action on $S^{-1}A$, and that $(S^G)^{-1}A^G \cong (S^{-1}A)^G$.

Solution. *Proof.* Let $G = \{\sigma_1, \dots, \sigma_n\}$ be a finite group of automorphisms of a ring A, and let A^G denote the subring of G-invariants. Let $a \in A$ and consider the monic polynomial

$$p(x) = \prod_{i=1}^{n} (x - \sigma_i(a)) \in A[x].$$

Since G is a subgroup, one of the σ_i's is the identity map, so $p(a) = 0$.

For $\sigma_j \in G$, we have

$$\sigma_j(p(x)) = \prod_{i=1}^{n}(x - \sigma_j(\sigma_i(a))) = \prod_{k=1}^{n}(x - \sigma_k(a)) = p(x).$$

So, we see that the coefficients of this polynomial are invariant under the action of G. So, a is integral over A^G. Since $a \in A$ is arbitrary, A is integral over A^G.

Let S be a multiplicatively closed subset of A such that $\sigma(S) \subseteq S$ for all $\sigma \in G$, and let $S^G = S \cap A^G$. We define an action σ' on $S^{-1}A$ by $\sigma'(a/s) = \sigma(a)/\sigma(s)$. Since $\sigma(S) \subseteq S$, the image is an element in $S^{-1}A$. Note that $\sigma'|_A = \sigma$.

We must show σ' is well-defined. If $a/s = a'/s'$ in $S^{-1}A$, then there exists $t \in S$ so that $t(as' - a's) = 0$. Then $0 = \sigma(t(as' - a's)) = \sigma(t)(\sigma(a)\sigma(s') - \sigma(a')\sigma(s)))$. Since $\sigma(s), \sigma(s'), \sigma(t) \in S$, this says $\sigma(a)/\sigma(s) = \sigma(a')/\sigma(s')$. So, σ' is well-defined.

Now we show that $(S^G)^{-1}A^G \cong (S^{-1}A)^G$. The canonical homomorphism $A \to S^{-1}A$ given by $a \mapsto a/1$ takes A^G to $(S^{-1}A)^G$, since both a and 1 lie in A^G. This gives us $\varphi : A^G \to (S^{-1}A)^G$. For $s \in S^G$, $\varphi(s) = s/1$, which is a unit in $(S^{-1}A)^G$. Thus, φ induces a homomorphism $\overline{\varphi} : (S^G)^{-1}A^G \to (S^{-1}A)^G$.

Let $a/s \in (S^G)^{-1}A^G$ lie in the kernel of $\overline{\varphi}$. Then $0 = \overline{\varphi}(a/s) = \varphi(a)/\varphi(s)$. Since this is zero in $(S^{-1}A)^G$. So, there exists $t \in S^G$ so that $t\varphi(a) = ta/1 = 0$. There exists $u \in S$ so that $uta = 0$ in A. But this says $a/s = 0$ in $(S^G)^{-1}A^G$

Mark, finish this. You need $u \in S^G$.

$\square$

Exercise 13. In the situation of Exercise 12, let $\mathfrak{p}$ be a prime ideal of A^G, and let P be the set of prime ideals of A whose contraction is $\mathfrak{p}$. Show that G acts transitively on P. In particular, P is *finite*.

[Let $\mathfrak{p}_1$, $\mathfrak{p}_2 \in P$ and let $x \in \mathfrak{p}_1$. Then $\Pi_\sigma \sigma(x) \in \mathfrak{p}_1 \cap A^G = \mathfrak{p} \subseteq \mathfrak{p}_2$, hence $\sigma(x) \in \mathfrak{p}_2$ for some $\sigma \in G$. Deduce that $\mathfrak{p}_1$ is contained in $\cup_{\sigma \in G}\sigma(\mathfrak{p}_2)$, and then apply Proposition 1.11 and Corollary 5.9.]

Proof. $\square$

Exercise 14. Let A be an integrally closed domain, K its field of fractions and L a finite normal separable extension of K. Let G be the Galois group of L over K and let B be the integral closure of A in L. Show that $\sigma(B) = B$ for all $\sigma \in G$, and that $A = B^G$.

Proof. $\square$

Exercise 15. Let A, K be as in Exercise 14, let L be any finite extension field of K, and let B be the integral closure of A in L. Show that, if $\mathfrak{p}$ is any prime ideal of A, then the set of prime ideals $\mathfrak{q}$ of B which contract to $\mathfrak{p}$ is finite (in other words, that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ has finite fibers).

[Reduce to the two cases (a) L separable over K and (b) L purely inseparable over K. In case (a), embed L in a finite normal separable extension of K, and use Exercises 13 and 14. In case (b), if $\mathfrak{q}$ is a prime ideal of B such that $\mathfrak{q} \cap A = \mathfrak{p}$, show that $\mathfrak{q}$ is the set of all $x \in B$ such that $x^{p^m} \in \mathfrak{p}$ for some $m \geq 0$, where p is the characteristic of K, and hence that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is bijective.

Solution. *Proof.* □

Exercise 16 (Noether's Normalization Lemma)**.** Let k be a field and let $A \neq 0$ be a finitely generated k-algebra. Then there exist elements $y_1, \ldots, y_r \in A$ which are algebraically independent over k and such that A is integral over $k[y_1, \ldots, y_k]$.

We shall assume that k is *infinite.* (The result is still true if k is finite, but a different proof is needed.) Let $x_1, \ldots, x_n$ generate A as a k-algebra. We can renumber the x_i so that $x_1, \ldots, x_r$ are algebraically independent over k and each of $x_{r+1}, \ldots, x_n$ is algebraic over $k[x_1, \ldots, x_r]$. Now proceed by induction on n. If $n = r$ there is nothing to do, so suppose $n > r$ and the result true for $n-1$ generators. The generator x_n is algebraic over $k[x_1, \ldots, x_{n-1}]$, hence there exists a polynomial $f \neq 0$ in n variables such that $f(x_1, \ldots, x_{n-1}, x_n) = 0$. Let F be the homogeneous part of highest degree in f. Since k is infinite, there exist $\lambda_1, \ldots, \lambda_{n-1} \in k$ such that $F(\lambda_1, \ldots, \lambda_{n-1}, 1) \neq 0$. Put $x'_i = x_i - \lambda_i x_n$ $(1 \leq i \leq n-1)$. Show that the x_n is integral over the ring $A'[x'_1, \ldots, x'_{n-1}]$, and hence that A is integral over A'. Then apply the inductive hypothesis to A' to complete the proof.

From the proof it follows that $y_1, \ldots, y_r$ may be chosen to be linear combinations of $x_1, \ldots, x_n$. This has the following geometrical interpretation: if k is algebraically closed and X is an affine algebraic variety in k^n with coordinate ring $A \neq 0$, then there exists a linear subspace L of dimension r in k^n and a linear mapping to k^n onto L which maps X *onto* L. [Use Exercise 2].

Solution. *Proof.* □

Exercise 17 (Nullstellensatz (weak form))**.** Let X be an affine algebraic variety in k^n, where k is an algebraically closed field, and let $I(X)$ be the ideal of X in the polynomial ring $k[t_1, \ldots, t_n]$. (Chapter 1, Exercise 27.) If $I(X) \neq (1)$ then X is not empty. [Let $A = k[t_1, \ldots, t_n]/I(X)$ be the coordinate ring of X. Then $A \neq 0$, hence by Exercise 16 there exists a linear subspace L of dimension ≥ 0 in k^n and a mapping of X *onto* L. Hence $X \neq \emptyset$.]

Deduce that every maximal ideal in the ring $k[t_1, \ldots, t_n]$ is of the form $(t_1 - a_1, \ldots, t_n - a_n)$.

Solution. *Proof.* □

Exercise 18. Let k be a field and let B be a finitely generated k-algebra. Suppose that B is a field. Then B is a finite algebraic extension of k. (This is another version of Hilbert's Nullstellensatz. The following proof is due to Zariski. For other proofs, see (5.24) and (7.9).)

Let $x_1, \ldots, x_n$ generate B as a k-algebra. Thye proof is by induction on n. If $n = 1$ the result is clearly true, so assume $n > 1$. Let $A = k[x_1]$ and let $K = k(x_1)$ be the field of fractions of A. By the inductive hypothesis, B is a finite algebraic extension of K, hence each of $x_2, \ldots, x_n$ satisfies a monic polynomial equation with coefficients in K, i.e. coefficients of the form a/b where a and b are in A. If f is the product of the denominators of all these coefficients, then each of $x_2, \ldots, x_n$ is integral over A_f. Hence B (and therefore K) is integral over A_f.

Suppose x_1 is transcendental over K. Then A is integrally closed, because it is a unique factorization domain. Hence A_f is integrally closed (**??**), and therefore $A_f = K$, which is clearly absurd. Hence x_1 is algebraic over k, hence K (and therefore B) is a finite extension of k.

Proof. □

Exercise 19. Deduce the result of Exercise 17 from Exercise 18.

Solution. *Proof.* □

Exercise 20. Let A be a subring of an integral domain B such that B is finitely generated over A. Show that there exists $s \neq 0$ in A and elements $y_1, \ldots, y_n$ in B, algebraically independent over A and such that B_s is integral over B'_s, where $B' = A[y_1, \ldots, y_n]$. [Let $S = A \setminus \{0\}$ and let $K = S^{-1}A$, the field of fractions of A. Then $S^{-1}B$ is a finitely generated K-algebra and therefore by the normalization lemma (Exercise 16) there exist $x_1, \ldots, x_n$ in $S^{-1}B$, algebraically independent over K and such that $S^{-1}B$ is integral over $K[x_1, \ldots, x_n]$. Let $z_1, \ldots, z_m$ generate B an an A-algebra. Then each z_j (regarded as an element of $S^{-1}B$) is integralover $K[x_1, \ldots, x_n]$. By writing an equation of integral dependence for each z_j, show that there exists $s \in S$ such that $x_i = y_i/s$ $(1 \leq i \leq n)$ with $y_i \in B$, and such that each sz_j is integral over B'. Deduce that this s satisfies the conditions stated.]

Solution. *Proof.* □

Exercise 21. Let A, B be as in Exercise 20. Show that there exists $s \neq 0$ in A such that, if Ω is an algebraically closed field and $f : A \to \Omega$ is a homomorphism for which $f(s) \neq 0$, then f can be extended to a homomorphism $B \to \Omega$. [With the notation of Exercise 20, f can be extended first of all to B', for example by mapping each y_i to 0; then to B'_s (because $f(s) \neq 0$), and finally to B_s (by Exercise 2, because B_s is integral over B'_s).]

Solution. *Proof.* □

Exercise 22. Let A, B be as in Exercise 20. If the Jacobson radical of A is zero, then so is the Jacobson radical of B. [Let $v \neq 0$ be an element of B. We have to show that there is a maximal ideal of B which does not contain v. By applying Exercise 21 to the ring B_v and its subring A, we obtain an element $s \neq 0$ in A. Let $\mathfrak{m}$ be a maximal ideal of A such that $s \notin \mathfrak{m}$, and let $k = A/\mathfrak{m}$. Then the canonical mapping $A \to k$ extends to a homomorphism g of B_v into an algebraic closure Ω of k. Show that $g(v) \neq 0$ and that $\operatorname{Ker}(g) \cap B$ is a maximal ideal of B.]

Solution. *Proof.* □

Exercise 23. Let A be a ring. Show that the following are equivalent:

i) Every prime ideal in A is an intersection of maximal ideals.

ii) In every homomorphic image of A the nilradical is equal to the Jacobson radical.

iii) Every prime ideal in A which is not maximal is equal to the intersection of the prime ideals which contain it strictly.

[The only hard part is (iii)⇒(i). Suppose (i) is false, then there is a prime ideal which is not an intersection of maximal ideals. Passing to the quotient ring, we may assume that A is an integral domain whose Jacobson radical $\mathfrak{R}$ is not zero. Let f be a non-zero element of $\mathfrak{R}$. Then $A_f \neq 0$, hence A_f has a maximal ideal, whose contraction in A is a prime ideal $\mathfrak{p}$ such that $f \notin \mathfrak{p}$, and which is maximal with respect to this property. Then $\mathfrak{p}$ is not maximal and is not equal to the intersection of the prime ideals strictly containing $\mathfrak{p}$.]

A ring A with the three equivalent properties above is called a *Jacobson ring.*

Solution. *Proof.* □

Exercise 24. Let A be a Jacobson ring (Exercise 23) and B an A-algebra. Show that if B is either (i) integral over A or (ii) finitely generated as an A-algebra, then B is Jacobson. [Use Exercise 22 for (ii).]

In particular, every finitely generated ring, and every finitely generated algebra over a field, is a Jacobson ring.

Solution. *Proof.* □

Exercise 25. Let A be a ring. Show that the following are equivalent:

i) A is a Jacobson ring;

ii) Every finitely generated A-algebra B which is a field is finite over A.

[(i)⇒(ii). Reduce to the case where A is a subring of B, and use Exercise 21 . If $s \in A$ is as in Exercise 21, then there exists a maximal ideal $\mathfrak{m}$ of A not containing s, and the homomorphism $A \to A/\mathfrak{m} = k$ extends to a homomorphism g of B into the algebraic closure of k. Since B is a field, g is injective, and $g(B)$ is algebraic over k, hence finite algebraic over k.

(ii)⇒(i). Use criterion (iii) of Exercise 23. Let $\mathfrak{p}$ be a prime ideal of A which is not maximal, and let $B = A/\mathfrak{p}$. Let f be a non-zero element of B. Then B_f is a finitely generated A-algebra. If it is a field it is finite over B, hence integral over B and therefore B is a field by (5.7). Hence B_f is not a field an therefore has a non-zero prime ideal, who contraction in B is a non-zero ideal $\mathfrak{p}'$ such that $f \notin \mathfrak{p}'$.]

Solution. *Proof.* □

Exercise 26. Let X be a topological space. A subset of X is *locally closed* if it is the intersection of an open set an a closed set, or equivalently if it is open in its closure.

The following conditions on a subset X_0 of X are equivalent:

(1) Every non-empty locally closed subset of X meets X_0;

(2) Fore very closed set E in X we have $\overline{E \cap X_0} = E$;

(3) The mapping $U \mapsto U \cap X_0$ of the collection of open sets of X onto the collection of opensets of X_0 is *bijective*.

A subset X_0 satisfying these conditions is said to be *very dense* in X.

If A is a ring, show that the following are equivalent:

i) A is a Jacobson ring;

ii) The set of maximal ideals of A is very dense in Spec(A);

iii) Every locally closed subset of Spec(A) consisting of a single point is closed. [ii) and iii) are geometrical formulations of conditions ii) and iii) of Exercise 23 .]

Solution. *Proof.* □

Exercise 27. Let A, B be two local rings. B is said to *dominate* A if A is a subring of B and the maximal ideal $\mathfrak{m}$ of A is contained in the maximal ideal $\mathfrak{n}$ of B (or, equivalent, if $\mathfrak{m} = A \cap \mathfrak{n}$). Ket J be a field and let Σ be the set of all local subrings of K. If Σ is ordered by the relation of domination, show that Σ has maximal elements and that $A \in \Sigma$ is maximal if and only if A is a valuation ring of K. [Use (**??**).]

Solution. *Proof.* □

Exercise 28. Let A be an integral domain, K its field of fractions. Show that the following are equivalent:

i) A is a valuation ring of K;

ii) if $\mathfrak{a}$, $\mathfrak{b}$, are any two ideals of A, then either $\mathfrak{a} \subset \mathfrak{b}$ or $\mathfrak{b} \subset \mathfrak{a}$.

Deduce that if A is a valuation ring and $\mathfrak{p}$ is a prime ideal of A, then $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are valuation rings of their fields of fractions.

Solution. *Proof.* □

Exercise 29. Let A be a valuation ring of a field K. Show that every subring of K which contains A is a local ring of A.

Solution. *Proof.* □

Exercise 30. Let A be a valution ring of a field K. The group U of units of A is a subgroups of the multiplicative group K^* of K.

Let $\Gamma = K^*/U$. If $\xi, \eta \in \Gamma$ are represented by $x, y \in K$, define $\xi \geq \eta$ to mean $xy^{-1} \in A$. Show that this defines a total ordering on Γ which is compatible with the group structure (i.e. $\xi \geq \eta \Rightarrow \xi\omega \geq \eta\omega$ for all $\omega \in \Gamma$). In other words, Γ is a totally ordered abelian group. It is called the *value group* of A.

Let $v : K^* \to \Gamma$ be the canonical homomorphism. Show that $v(x + y) \geq \min(v(x), v(y))$ for all $x, y \in K^*$.

Solution. *Proof.* □

Exercise 31. Conversely, let Γ be a totally ordered abelian group (written *additively*), and let K be a field. A *valuation of K with values in* Γ is a mapping $v : K^* \to \Gamma$ such that

1) $v(xy) = v(x) + v(x)$

2) $v(x + y) \geq \min(v(x), v(y))$, for all $x, y \in K^*$.

   Show that the set of elements of $x \in K^*$ such that $v(x) \geq 0$ is a valuation ring of K. This ring is called the *valuation ring* of v, and the subgroup $v(K^*)$ of Γ is the *value group* of v.

   Thus the concepts of valuation ring and valuation are essentially equivalent.

Solution. *Proof.* □

Exercise 32. Let Γ be a totally ordereed abelian group. A subgroup Δ of Γ is *isolated* in Γ if, whenever $0 \leq \beta \leq \alpha$ and $\alpha \in \Delta$, we have $\beta \in \Delta$. Let A be a valuation ring of a field K, with vale group Γ (Exercise 31.) If $\mathfrak{p}$ is a prime ideal of A, show that $v(A \setminus \mathfrak{p})$ is the set of element ≥ 0 in an isolated subgroup Δ of Γ, and that the mapping so defined of $\operatorname{Spec}(A)$ into the set of isolated subgroups of Γ is bijective.

If $\mathfrak{p}$ is prime ideal of A, what are the value groups of the valuation rings $A/\mathfrak{p}$, $A_{\mathfrak{p}}$

Solution. *Proof.* □

Exercise 33. Let Γ be a totally ordered abelian group. We shall show how to construct a field K and a valuation v of K with Γ as value group. Let k be any field and let $A = k[\Gamma]$ be the group algebra of Γ over k. By definition, A is freely generated as a k-vector space by the elements x_α ($\alpha \in \Gamma$) such that $x_\alpha x_\beta = x_{\alpha+\beta}$. Show that A is an integral domain.

If $u = \lambda_1 x_{\alpha_1} + \cdots + \lambda_n x_{\alpha_n}$ is any non-zero element of A, where the λ_i are all $\neq 0$ and $\alpha_1 < \cdots < \alpha_n$, define $v(u) = \alpha_1$. Show that the mapping $v_0 : A \setminus \{0\} \to \Gamma$ satisfies condition (1) and (2) of Exercise 31.

Let K be the field of fractions of A. Show that v_0 be be uniquely extended to a valuation v of K, and that the value group of v is precisely Γ.

Solution. *Proof.* □

Exercise 34. Let A be a valuation ring and K its field of fractions. Lt $f : A \to B$ be a ring homomorphism such that $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a *closed* mapping. Then if $g : B \to K$ is any A-algebra homomorphism (i.e., if $g \circ f$ is the embedding of A in K) we have $g(B) = A$.

[Let $C = g(B)$; obviously $C \supseteq A$. Let $\mathfrak{n}$ be a maximal ideal of C. Since f^* is closed, $\mathfrak{m} = \mathfrak{n} \cap A$ is the maximal ideal of A, when $A_{\mathfrak{m}} = A$. Also the local ring $C_{\mathfrak{n}}$ dominates $A_{\mathfrak{m}}$. Hence by Exercise 27 we have $C_{\mathfrak{n}} = A$ and therefore $C \subseteq A$.]

Solution. *Proof.* □

Exercise 35. From Exercises 1 and 3 it folows that, if $f : A \to B$ is integral and C is any A-algebra, then the mapping $(f \otimes 1)^* : \operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec}(C)$ is a closed map.

Conversely, supose that $f : A \to B$ has this property and that B is an itegral domain. Then f is integral. [Replacing A by its image in B, reduce to the case where $A \subseteq B$ and f is the injection. Let K be the field of fractions of B and let A' be a valuation ring of K containing A. By (5.22) it is enough to show that A' contains B. By hypothesis $\operatorname{Spec}(B \otimes_A A') \to \operatorname{Spec}(A')$ is a closed map. Apply the result of Exercise 34 to the homomorphism $B \otimes_A A' \to K$ defined by

$b \otimes a' \mapsto ba'$. It follows that $ba' \in A$ for all $b \in B$ and all $a' \in A'$; taking $a' = 1$, we have what we want.]

Show that the result just proved remains valid if B is a ring with only finitely many minimal prime ideal (e.g., if B is Noetherian). [Let $\mathfrak{p}_i$ be the minimal prime ideals. Then each composite homomorphism $A \to B \to B/\mathfrak{p}_i$ is integral, hence $A \to \prod(B/\mathfrak{p}_i)$ is integral, hence $A \to B/\mathfrak{N}$ is integral (where $\mathfrak{N}$ is the nilradical of B), hence finally $A \to B$ is integral.]

Solution. *Proof.* □

Chapter 6

Chain Conditions

Exercises

Exercise 1. i) Let M be a Noetherian A-module and $u : M \to M$ a module homomorphism. If u is surjective, then u is an isomorphism.

ii) If M is Artinian and u is injective, then again u is an isomorphism.

[For (i), consider the submodules $\operatorname{Ker}(u^n)$; for (ii) the quotient modules $\operatorname{Coker}(u^n)$.]

Solution. i) *Proof.* Let M be a Noetherian A-module and $u : M \to M$ a surjective module homomorphism.

Consider the chain of submodules of M

$$\operatorname{Ker}(u) \subset \operatorname{Ker}\left(u^2\right) \subset \operatorname{Ker}\left(u^3\right) \subset \operatorname{Ker}\left(u^4\right) \subset \cdots$$

Since M is Noetherian, this chain must be stationary. So, there exists some $n \in \mathbb{N}$ so that $\operatorname{Ker}(u^n) = \operatorname{Ker}\left(u^{n+1}\right)$.

Suppose $x \in \operatorname{Ker}(u)$. We remark that since u is surjective, so is u^k for all $k \in \mathbb{N}$, so u^n is surjective. Since u^n is surjective, there exists $y \in M$ so that $u^n(y) = x$. Since $x \in \operatorname{Ker}(u)$, we have $0 = u(x) = u(u^n(y)) = u^{n+1}(y)$. Thus, $y \in \operatorname{Ker}\left(u^{n+1}\right) = \operatorname{Ker}(u^n)$. Hence, we see that $x = u^n(y) = 0$. Thus, u is also injective.

This shows u is an isomorphism. □

ii) *Proof.* Suppose M is an Artinian A-module and $u : M \to M$ a injective module homomorphism.

Consider the chain of submodules of M

$$\operatorname{Coker}(u) \supset \operatorname{Coker}\left(u^2\right) \supset \operatorname{Coker}\left(u^3\right) \supset \operatorname{Coker}\left(u^4\right) \supset \cdots$$

Since M is Artinian, this chain must be stationary. So, there exists some $n \in \mathbb{N}$ so that $\operatorname{Coker}(u^n) = \operatorname{Coker}\left(u^{n+1}\right)$. Since $\operatorname{Coker}(u^n) = \operatorname{Coker}\left(u^{n+1}\right)$, we must have $u^n(M) = u^{n+1}(M)$.

Let $y \in M$ and consider $u^n(y)$. Since $u^n(M) = u^{n+1}(M)$, there exists $x \in M$ so that $u^{n+1}(x) = u^n(y)$. Then $u^n(u(x)) = u^n(y)$. Since u is injective, so is u^k for all $k \in \mathbb{N}$. Hence u^n is injective, so we see that $u(x) = y$. Since $y \in M$ is arbitrary, this shows u is surjective as well.

Hence u is an isomorphism. □

Exercise 2. Let M be an A-module. If every non-empty set of finitely generated submodules of M has a maximal element, then M is Noetherian.

Solution. *Proof.* Let M be an A-module and suppose that every non-empty set of finitely generated submodules of M has a maximal element.

Let $N \subset M$ be any A-submodule of M. Let Σ be the collection of all finitely generated submodules of N. Since submodules of N are also submodules of M, by hypothesis Σ has a maximal element, N'.

Choose $x \in N$. Then the module generated by N' and $\{x\}$ is finitely generated, so by maximality, this module is N'. But this says that $x \in N'$. Since $x \in N$ is arbitrary, it follows that $N = N'$. So, N is finitely generated.

Since $N \subset M$ is an arbitrary A-submodule of M, M is Noetherian by definition. □

Exercise 3. Let M be an A-module and let N_1, N_2 be submodules of M. If M/N_1 and M/N_2 are Noetherian, so is $M/(N_1 \cap N_2)$. Similarly with Artinian in place of Noetherian.

Solution. *Proof.* Let M be an A-module and let N_1, N_2 be submodules of M. Suppose M/N_1 and M/N_2 are Noetherian. □

Exercise 4. Let M be a Noetherian A-module and let $\mathfrak{a}$ be the annihilator of M in A. Prove that $A/\mathfrak{a}$ is a Noetherian ring.

If we replace "Noetherian" by "Artinian", is it still true?

Solution. *Proof.* Let M be a Noetherian A-module and let $\mathfrak{a}$ be the annihilator of M in A.

□

Exercise 5. A topological space X is said to be *Noetherian* if the open subsets of X satisfy the ascending chain condition (or, equivalently, the maximal condition). Since closed subsets are complements of open subsets, it comes to the same thing to say that the closed subsets of X satisfy the descending chain condition (or, equivalently, the minimal condition). Show that, if X is Noetherian, then every subspace of X is Noetherian, and that X is quasi-compact.

Solution. *Proof.* Let X be a Noetherian topological space and let $Y \subseteq X$ be a subspace. Let $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$ be a chain open subsets of Y. Since Y is a subspace, each V_i is of the form $Y \cap U_i$ for an open set U_i in X. Since X is a Noetherian topological space, the chain $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ is eventually stationary. It then follows that the chain $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$ is also eventually stationary. Since this chain is arbitrary, Y is Noetherian. Since Y is arbitrary, every subspace of X is Noetherian.

Let $\mathscr{U}$ be an open cover of X. Assume $\mathscr{U}$ has no finite subcover. Choose $U_1 \in \mathscr{U}$. Since $\mathscr{U}$ has no finite subcover, U_1 doesn't cover X, so there exists $U_2 \in \mathscr{U}$ so that $U_1 \subsetneq U_1 \cup U_2$. Suppose $U_1, \ldots, U_n$ are in $\mathscr{U}$ so that

$$U_1 \subsetneq U_1 \cup U_2 \subsetneq U_1 \cup U_2 \cup U_3 \subsetneq \cdots \subsetneq U_1 \cup \cdots \cup U_n$$

Since $\mathscr{U}$ has no finite subcover, $\{U_1, \ldots, U_n\}$ doesn't cover X, so there exists $U_{n+1} \in \mathscr{U}$ so that $U_1 \cup \cdots \cup U_n \subsetneq U_1 \cup \cdots \cup U_n \cup U_{n+1}$. This produces a chain

$$U_1 \subsetneq U_1 \cup U_2 \subsetneq U_1 \cup U_2 \cup U_3 \subsetneq \cdots \subsetneq U_1 \cup \cdots \cup U_n \subsetneq \cdots .$$

Since X is Noetherian, this chain of open sets must be stationary. This is a contradiction. So, $\mathscr{U}$ has a finite subcover and since $\mathscr{U}$ is an arbitrary open cover of X, X is quasi-compact. □

Exercise 6. Prove the following are equivalent:

i) X is Noetherian

ii) Every open subspace of X is quasi-compact.

iii) Every subspace of X is quasi-compact.

Solution. *Proof.* i)⇒ii) Suppose X is Noetherian and let Y be an open subspace of X. By Exercise 5, Y is also Noetherian, and by Exercise 5 again, Y is quasi-compact. Since Y be an arbitrary open subspace of X, every open subspace of X is quasi-compact.

ii)⇒iii) Suppose every open subspace of X is quasi-compact. Let Y be any subspace of X. Let $\mathscr{U}$ be any open cover of Y. Then $Z = \bigcup_{U \in \mathscr{U}} U$ is an open subspace of X, so it's quasi-compact. So, there is a subcover U_1, $\ldots$, U_n that covers Z. However, Z contains Y, so the collection U_1, $\ldots$, U_n also covers Y. So, $\mathscr{U}$ has a finite subcover of Y. Since $\mathscr{U}$ is arbitrary, every Y is quasi-compact, and since Y is an arbitrary subspace of X, every subspace of X is quasi-compact.

iii)⇒i) Suppose every subspace of X is quasi-compact. Let

$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots \subseteq U_n \subseteq \cdots \tag{6.1}$$

be a chain of open sets in X. Then $Y = \cup_{i=1}^{\infty} U_i$ is a subspace of X. By hypothesis, Y is quasi-compact, and it follows that the chain (6.1) is stationary.

□

Exercise 7. A Noetherian space is a finite union of irreducible closed subspaces. [Consider the set Σ of closed subsets of X which are not finite unions of irreducible closed subspaces.] Hence the set of irreducible components of a Noetherian space is finite.

Solution. *Proof.* Let X be a Noetherian space and let Σ be the set of closed subsets of X which are not finite unions of irreducible closed subspaces. Suppose Σ is not empty. By the minimal property, Σ contains a smallest element, F. Then F is not a finite union of irreducible closed subspaces, so in particular, F is not irreducible. So, $F = F_1 \cup F_2$, where F_1, F_2 are proper, closed subsets of F. By the minimality of F, F_1 and F_2 are finite unions of irreducible closed subspaces. But then so is F, a contradiction. So, Σ is empty and X itself is a finite union of irreducible closed subspaces. Hence the set of irreducible components of a Noetherian space is finite. $\square$

Exercise 8. If A is a Noetherian ring then $\mathrm{Spec}(A)$ is a Noetherian topological space. Is the converse true?

Solution. *Proof.* Let A be a Noetherian ring. Let $V(\mathfrak{p}_1) \supset V(\mathfrak{p}_2) \supset V(\mathfrak{p}_3) \supset \cdots$ be a nested sequence of closed subsets of A. Then $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \mathfrak{p}_3 \subset \cdots$ is an increasing nested sequence of prime ideals in A. Since A is Noetherian, this sequence is stationary. So, eventually $\mathfrak{p}_N = \mathfrak{p}_n$ for all $n \geq N$. But then $V(\mathfrak{p}_N) = V(\mathfrak{p}_n)$ for all $n \geq N$. That is, the chain $V(\mathfrak{p}_1) \supset V(\mathfrak{p}_2) \supset V(\mathfrak{p}_3) \supset \cdots$ is stationary. It follows that $\mathrm{Spec}(A)$ is a Noetherian topological space. $\square$

Find a counter example.

Exercise 9. Deduce from Exercise 8 that the set of minimal prime ideals in a Noetherian ring is finite.

Solution. *Proof.* Let A be a Noetherian ring. By Exercise 8, $\mathrm{Spec}(A)$ is a Noetherian topological space. By Exercise 7, $\mathrm{Spec}(A)$ is a finite union of irreducible closed subspaces.

$\square$

Exercise 10. If M is a Noetherian module (over an arbitrary ring A) then Supp(M) is a closed Noetherian subspace of $\mathrm{Spec}(A)$.

Solution. *Proof.* $\square$

Exercise 11. Let $f : A \to B$ be a ring homomorphism and suppose that $\mathrm{Spec}(B)$ is a Noetherian space (Exercise 5). Prove that $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a closed mapping if and only if f has the going-up property (Chapter 5, Exercise 10).

Solution. *Proof.* □

Exercise 12. Let A be a ring such that $\mathrm{Spec}(A)$ is a Noetherian space. Show that the set of prime ideals of A satisfies the ascending chain condition. Is the converse true?

Solution. *Proof.* □

Chapter 7

Noetherian Rings

Exercises

Exercise 1. Let A be a non-Noetherian ring and let Σ be the set of ideals in A which are not finitely generated. Show that Σ has maximal elements and that the maximal elements of Σ are prime ideals.
[Let $\mathfrak{a}$ be a maximal element of Σ, and suppose that there exist $x, y \in A$ such that $x \notin \mathfrak{a}$ and $y \notin \mathfrak{a}$ and $xy \in \mathfrak{a}$. Show that there exists a finitely generated ideal $\mathfrak{a}_0 \subseteq \mathfrak{a}$ such that $\mathfrak{a}_0 + (x) = \mathfrak{a} + (x)$, and that $\mathfrak{a} = \mathfrak{a}_0 + x \cdot (\mathfrak{a} : x)$. Since $(\mathfrak{a} : x)$ strictly contains $\mathfrak{a}$, it is finitely generated and therefore so is $\mathfrak{a}$.]

Solution. *Proof.* Let A be a non-Noetherian ring and let Σ be the set of ideals in A which are not finitely generated. The set Σ clearly satisfies Zorn's lemma, so Σ has maximal elements.

Let $\mathfrak{a}$ be a maximal element of Σ, and suppose that there exist $x, y \in A$ such that $x \notin \mathfrak{a}$ and $y \notin \mathfrak{a}$ and $xy \in \mathfrak{a}$. Since $\mathfrak{a} + (x)$ properly contain $\mathfrak{a}$, it must be finitely generated. Any finite set of generators for $\mathfrak{a} + (x)$ must include x since $x \notin \mathfrak{a}$. Choose a finite set of generators $\{a_1, \ldots, a_n, x\}$ for $\mathfrak{a} + (x)$. Similarly, choose a finite set of generators $\{b_1, \ldots, b_m, y\}$ for $\mathfrak{a} + (y)$.

Mark, finish this one.

□

Exercise 2. Let A be a Noetherian ring and let $f = \sum_{n=0}^{\infty} a_n x^n \in [[A]]$. Prove that f is nilpotent if and only if each a_n is nilpotent.

Solution. *Proof.* Let A be a Noetherian ring and let $f = \sum_{n=0}^{\infty} a_n x^n \in [[A]]$. We note that $A[[x]]$ is Noetherian since A is.

By Exercise 5(ii) in Chapter 1, if f is nilpotent, then a_n is nilpotent for all $n \geq 0$.

Suppose a_n is nilpotent for all $n \geq 0$. □

Exercise 3. Let $\mathfrak{a}$ be an irreducible ideal in a ring A. Then the following are equivalent:

i) $\mathfrak{a}$ is primary;

ii) for every multiplicatively closed subset S of A we have $(S^{-1}\mathfrak{a})^c = (\mathfrak{a} : x)$ for some $x \in S$;

iii) the sequence $(\mathfrak{a} : x^n)$ is stationary, for every $x \in A$.

Solution. *Proof.* i) $((i) \Rightarrow (ii))$. Suppose $\mathfrak{a}$ is primary. Let S be a multiplicatively closed subset of A and let $a/b \in (S^{-1}\mathfrak{a})^c$. Then $a/b = c/1$ for some $c \in \mathfrak{a}$, and hence there exists $x \in S$ so that $x(a - bc) = 0$. So, $xbc = xa \in \mathfrak{a}$. But then $c \in (\mathfrak{a} : x)$ ...

Mark, start here.

ii) $((ii) \Rightarrow (iii))$

iii) $((iii) \Rightarrow (i))$

□

Exercise 4. Which of the following rings is Noetherian?

i) The ring of rational functions of z having no pole on the circle $|z| = 1$.

ii) The ring of power series in z with a positive radius of convergence.

iii) The ring of power series in z with an infinite radius of convergence.

iv) The ring of polynomials in z whose first k derivatives vanish at the origin (k being a fixed integer).

v) The ring of polynomials in z, w all of whose partial derivatives with respect to w vanish for $= 0$.

In all cases, the coefficients are complex numbers.

Solution. *Proof.* □

Exercise 5. Let A be a Noetherian ring, B a finitely generated A-algebra, G a finite group of A-automorphisms of B, and B^G the set of all elements of B which are left fixed by every element of G. Show that B^G is a finitely generated A-algebra.

Solution. *Proof.* □

Exercise 6. If a finitely generated ring K is a field, it is a finite field. [If K has characteristic 0, we have $\mathbb{Z} \subset \mathbb{Q} \subseteq K$. Since K is finitely generated over $\mathbb{Z}$ it is finitely generated over $\mathbb{Q}$, hence by (**??**) is a finitely generated $\mathbb{Q}$-module. Now apply (**??**) to obtain a contradiction. Hence K is of characteristic $p > 0$, hence if finitely generated as a $\mathbb{Z}/(p)$-algebra. Use (**??**) to complete the proof.]

Solution. *Proof.* □

Exercise 7. Let X be an affine algebraic variety given by a family of equations $f_\alpha(t_1, \ldots, t_n) = 0 \quad (\alpha \in I)$. (Chapter 1, Exercise 27). Show that there exists a finite subset I_0 of I such that X if given by the equations $f_\alpha(t_1, \ldots, t_n) = 0 (\alpha \in I_0)$.

Proof. □

Exercise 8. If $A[x]$ is Noetherian, is A necessarily Noetherian?

Solution. *Proof.* □

Exercise 9. Let A be a ring such that

(1) for each maximal ideal $\mathfrak{m}$ of A, the local ring $A_\mathfrak{m}$ is Noetherian;

(2) for each $x \neq 0$ in A, the set of maximal ideals of A which contains x is finite.

Show that A is Noetherian.

[Let $\mathfrak{a} \neq 0$ be an ideal in A. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals which contain $\mathfrak{a}$. Choose $x_0 \neq 0$ in $\mathfrak{a}$ and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_{r+s}$ be the maximal ideals which contain x_0. Since $\mathfrak{m}_{r+1}, \ldots, \mathfrak{m}_{r+s}$ do not contain $\mathfrak{a}$, there exist $x_j \in \mathfrak{a}$ such that $x_j \notin \mathfrak{m}_{r+j}$, $(1 \leq j \leq s)$. Since each $A_{\mathfrak{m}_i}$ $(1 \leq i \leq r)$ is Noetherian, the extension of $\mathfrak{a}$ in $A_{\mathfrak{m}_i}$ is finitely generated. Hence there exist $x_{s+1}, \ldots, x_t$ in $\mathfrak{a}$ whose images in $A_{\mathfrak{m}_i}$ generate $A_{\mathfrak{m}_i}\mathfrak{a}$ for $i = 1, \ldots, r$. Let $\mathfrak{a}_0 = (x_0, \ldots, x_t)$. Show that $\mathfrak{a}_0$ and $\mathfrak{a}$ have the same extension in $A_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$, and deduce by (**??**) that $\mathfrak{a}_0 = \mathfrak{a}$.]

Solution. *Proof.* □

Exercise 10. Let M be Noetherian A-module. Show that $M[x]$ (Chapter 2, Exercise 6) is a Noetherian $A[x]$-module.

Solution. *Proof.* □

Exercise 11. Let A be a ring such that each local ring $A_\mathfrak{p}$ is Noetherian. Is A necessarily Noetherian?

Solution. *Proof.* □

Exercise 12. Let A be a ring and B a faithfully flat A-algebra (Chapter 3, Exercise 16). If B is Noetherian, show that A is Noetherian. [Use the ascending chain condition.]

Solution. *Proof.* □

Exercise 13. Let $f : A \to B$ be a ring homomorphism of finite type and let $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be the mapping associated with f. Show that the fibers of f^* are Noetherian subspaces of B.

Solution. *Proof.* □

Exercise 14. Let k be an algebraically closed field, let A denote the polynomial ring $k[t_1, \ldots, t_n]$ and let $\mathfrak{a}$ be an ideal in A. Let V be the variety in k^n defined by the ideal $\mathfrak{a}$, so that V is the set of all $x = (x_1, \ldots, x_n) \in k^n$ such that $f(x) = 0$ for all the $f \in \mathfrak{a}$. Let $I(V)$ be the ideal of V, i.e. the ideal of all polynomials $g \in A$ such that $g(x) = 0$ for all $x \in V$. Then $I(V) = \operatorname{rad}(\mathfrak{a})$.

[It is clear that $\operatorname{rad}(\mathfrak{a}) \subset I(V)$. Conversely, let $f \notin \operatorname{rad}(\mathfrak{a})$, then there is a prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ such that $f \notin \mathfrak{p}$. Let $\overline{f}$ be the image of f in $B = A/\mathfrak{p}$, let $C = B_f = B[1/\overline{f}]$, and let $\mathfrak{m}$ be a maximal ideal of C. Since C is a finitely generated k-algebra we have $C/\mathfrak{m} \cong k$, by (**??**). The images x_i in $C/\mathfrak{m}$ of the generators t_i, of A thus define a point $x = (x_1, \ldots, x_n) \in k^n$, and the construction shows that $x \in V$ and $f(x) \neq 0$.]

Solution. *Proof.* □

Exercise 15. Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal and k its residue field, and let M be a finitely generated A-module. Then the following are equivalent:

i) M is free;

ii) M is flat;

iii) the mapping of $\mathfrak{m} \otimes M$ into $A \otimes M$ is injective;

iv) $\operatorname{Tor}_1^A(k, M) = 0$.

[To show that iv)⇒i), let $x_1, \ldots, x_n$ be elements of M whose images in $M/\mathfrak{m}M$ form a k-basis of this vector space. By (**??**), the x_i generate M. Let F be the free A-module with basis $e_1, \ldots, e_n$ and define $\phi : F \to M$ by $\phi(e_i) = x_i$. Let $E = \operatorname{Ker}(\phi)$. Then the exact sequence $0 \to E \to F \to M \to 0$ gives us an exact sequence

$$0 \to k \otimes_A E \to k \otimes_A F \to k \otimes_A M \to 0.$$

Since $k \otimes F$ and $k \otimes M$ are vector sapces of the same dimension over k, it follows that $1 \otimes \phi$ is an isomorphism, hence $k \otimes E = 0$, hence $E = 0$ by Nakayama's lemma (E is finitely generated because it is a submodule of F, an A is Noetherian).]

Solution. *Proof.* □

Exercise 16. Let A be a Noetherian ring, M a finitely generated A-module. Then the following are equivalent:

i) M is free;

ii) M is flat;

iii) the mapping of $\mathfrak{m} \otimes M$ into $A \otimes M$ is injective;

iv) $\mathrm{Tor}_1^A(k, M) = 0$.

Solution. *Proof.* □

Exercise 17. Let A be a ring and M a Noetherian A-module. Show (by imitating the proofs of (**??**) and (**??**)) that every submodule N of M has a primary decomposition (Chapter 4, Exercises 20–23).

Solution. *Proof.* □

Exercise 18. Let A be a Noetherian ring, $\mathfrak{p}$ a prime ideal of A, and M a finitely generated A-module. Show that the following are equivalent:

i) $\mathfrak{p}$ belongs to 0 in M;

ii) there exists $x \in M$ such that $\mathrm{Ann}\,(x) = \mathfrak{p}$;

iii) there exists a submodule of M isomorphic to $A/\mathfrak{p}$.

Deduce that there exist a chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

suc that each quotient M_i/M_{i-1} is of the form $A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal of A.

Solution. *Proof.* □

Exercise 19. Let $\mathfrak{a}$ be an ideal in a Noetherian ring A. Let

$$\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{b}_i = \bigcap_{j=1}^{s} \mathfrak{c}_j.$$

be two minimal decompositions of $\mathfrak{a}$ as intersection of *irreducible* ideals. Prove that $r = s$ and that (possibly after re-indexing the $\mathfrak{c}_j$) $\operatorname{rad}(\mathfrak{b}_\mathfrak{i}) = \operatorname{rad}(\mathfrak{c}_\mathfrak{i})$ for all i.

[Show that for each $i = 1, \ldots, r$ there exists j such that

$$\mathfrak{a} = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_{i-1} \cap \mathfrak{c}_j \cap \mathfrak{b}_{i+1} \cap \cdots \cap \mathfrak{b}_r.]$$

Stat and prove an analogous result for modules.

Solution. *Proof.* □

Exercise 20. Let X be a topological space and let $\mathscr{F}$ be the smallest collection of subsets of X which contains all open subsets of X and is closed with respect to the formation of finite intersections and complements.

i) Show that a subset E of X belongs to $\mathscr{F}$ if and only if E is a finite union of sets of the form $U \cap C$, where U is open and C is closed.

ii) Suppose that X is irreducible and let $E \in \mathscr{F}$. Show that E is dense in X (i.e. that $\overline{E} = X$) if and only if E contains a non-empty closed set $X_0 \subseteq X$.

Solution. *Proof.* □

Exercise 21. Let X be a Noetherian topological space (Chapter 6, Exercise 5) and let $E \subseteq X$. Show that $E \in \mathscr{F}$ if and only if, for each irreducible closed set $X_0 \subseteq X$, either $\overline{E \cap X_0} \neq X_0$ or else $E \cap X_0$ contains a non-empty open subset of X_0. [Suppose $E \notin \mathscr{F}$. Then the collection of closed sets $X' \subseteq X$ such that $E \cap X' \notin \mathscr{F}$ is not empty and therefore has a minimal element X_0. Show that X_0 is irreducible and then that each of the alternatives above leads to the conclusion that $E \cap X_0 \in \mathscr{F}$.] The sets belonging $\mathscr{F}$ are called the *constructible subsets* of X.

Solution. *Proof.* □

Exercise 22. Let X be a Noetherian topological space and let E bee a subset of X. Show that E is open in X if and only if, for each irreducible closed subset X_0 in X, either $E \cap X_0 = \emptyset$ or else $E \cap X_0$ contains contains a non-empty open subset of X_0. [The proof is similar to that of Exercise 21.]

Solution. *Proof.* □

Exercise 23. Let A be a Noetherian ring, $f : A \to B$ a ring homomorphism of finite type (so that B is Noetherian). Let $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and let $f^* : Y \to X$ be the mapping associated with f. Then the image under f^* of a constructible subset E of Y is a constructible subset of X.

[By Exercise 20 it is enough to take $E = U \cap C$ where U is open and C is closed in Y; then, replacing B by a homomorphic image, we reduce to the case where E is open in Y. Since Y is Noetherian, E is quasi-compact and therefore a finite union of open sets of the form $\mathrm{Spec}(B_g)$. Hence reduce to the case $E = Y$. To show that $f^*(Y)$ is constructible, use the criterion of Exercise 21. Let X_0 be an irreducible closed subset of X such that $f^*(Y) \cap X_0$ is dense in X_0. We have $f^*(Y) \cap X_0 = f^*(f^{*-1}(X_0))$, and $f^{*-1}(X_0) = \mathrm{Spec}((A/\mathfrak{p}) \otimes_A B)$, where $X_0 = \mathrm{Spec}(A/\mathfrak{p})$. Hence to reduce to the case where A is an integral domain and f is injective. If $Y_1, \dots, Y_n$ are irreducible components of Y, it is enough to show that some f^*Y_i contains a non-empty open set inf X. So finally we are brought down to the situation in which A, B are integral domains and f is injective (and still of finite type); now use Chapter 5, Exercise 21 to complete the proof.]

Solution. *Proof.* □

Exercise 24. With the notation and hypotheses of Exercise 23, f^* is an open mapping $\Leftrightarrow$ f has the going-down property (Chapter 5, Exercise **??**). [Suppose f has the going-down property. As in Exercise 23, reduce to proving that $E = f^*(Y)$ is open in X. The going-down property asserts that if $\mathfrak{p} \in E$ and $\mathfrak{p}' \subseteq \mathfrak{p}$, then $\mathfrak{p}' \in E$: in other words, that if X_0 is an irreducible closed subset of X and X_0 meets E, then $E \cap X_0$ is dense in X_0. By Exercises 20 and 22, E is open in X.]

Solution. *Proof.* □

Exercise 25. Let A be Noetherian, $f : A \to B$ of finite type and *flat* (i.e., B is flat as an A-module). Then $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an open mapping. [Exercise 24 and Chapter 5, Exercise **??**.]

Solution. *Proof.* □

Grothendieck groups

Exercise 26. Let A be a Noetherian ring and let $F(A)$ denote the set of all isomorphism classes of finitely generated A-modules. Let C be the free abelian group generated by $F(A)$. With each short exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated A-modules we associate the element $(M') - (M) + (M'')$ of C, where (M) is the isomorphism class of M, etc. Let D be the subgroup of C generated by these elements, for all short exact sequences. The quotient group C/D is called the *Grothendieck group* of A, and is denoted by $K(A)$. If M is a finitely generated A-module, let $\gamma(M)$, or $\gamma_A(M)$, denote the image of (M) in $K(A)$.

i) Show that $K(A)$ has the following universal property: for each additive function λ on the class of finitely generated A-modules with values in an abelian group G, there exists a unique homomorphism $\lambda_0 : K(A) \to G$ such that $\lambda(M) = \lambda_0(\gamma(M))$.

ii) Show that $K(A)$ is generated by the element $\gamma(A/\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of A. [Use Exercise 18.]

iii) If A is a field, or more generally if A is a principal ideal domain, then $K(A) \cong \mathbb{Z}$.

iv) Let $f : A \to B$ be a *finite* ring homomorphism. Show that restriction of scalars gives rise to a homomorphism $f_! : K(B) \to K(A)$ such that $f_!(\gamma_B(N)) = \gamma_A(N)$ for a B-module N. If $g : B \to C$ is another finite ring homomorphism, show that $(g \circ f)_! = f_! \circ g_!$.

Solution. *Proof.* □

Exercise 27. Let A be a Noetherian ring and let $F_1(A)$ be the set of all isomorphism classes of finitely generated *flat* A-modules. Repeating the construction of Exercise 26 we obtain a group $K_1(A)$. Let $\gamma_1(M)$ denote the image of (M) in $K_1(A)$.

i) Show that tensor product of modules over A induces a commutative ring structure on $K_1(A)$, such that $\gamma_1(M) \cdot \gamma_1(N) = \gamma_1(M \otimes N)$. The identity element of this ring is $\gamma_1(A)$.

ii) Show that tensor product induces a $K_1(A)$-module structure on the group $K(A)$, such that $\gamma_1(M) \cdot \gamma(N) = \gamma(M \otimes N)$.

iii) If A is a (Noetherian) local ring, then $K_1(A) \cong \mathbb{Z}$.

iv) Let $f : A \to B$ be a ring homomorphism, B being Noetherian. Show that extension of scalars gives rise to a ring homomorphism $f^! : K_1(A) \to K_1(B)$ such that $f^!(\gamma_1(M)) = \gamma_1(B \otimes_A M)$. [If M is flat and finitely generated over A, then $B \otimes_A M$ is flat and finitely generated over B.] If $g : B \to C$ is another ring homomorphism (with C Noetherian), then $(f \circ g)^! = f^! \circ g^!$.

v) If $f : A \to B$ is a finite ring homomorphism then

$$f_!(f^!(x)y) = xf_!(y)$$

for $x \in K_1(A)$, $y \in K(B)$. In other words, regarding $K(B)$ as a $K_1(A)$-module by restriction of scalars, the homomorphism $f_!$ is a $K_1(A)$-module homomorphism.

Remark 1. Since $F_1(A)$ is a subset of $F(A)$ we have a group homomorphism $\epsilon : K_1(A) \to K(A)$, given the $\epsilon(\gamma_1(M)) = \gamma(M)$. If the ring A is finite-dimensional and *regular*, i.e., if all its local rings $A_\mathfrak{p}$ are regular (Chapter 11), it can be shown that ϵ is an isomorphism.

Solution. *Proof.* □

Chapter 8

Artin Rings

Exercises

Exercise 1. Let $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n = 0$ be a minimal primary decompostion of the zero ideal in a Noetherian ring, and let $\mathfrak{q}_i$ be $\mathfrak{p}_i$-primary. Let $\mathfrak{p}_i^{(r)}$ be the rth *symbolic power* of $\mathfrak{p}_i$ (Chapter 4, Exercise 13). Show that for each $i = 1, \ldots n$ there exists an integer r_i such that $\mathfrak{p}_i^{(r_i)} \subseteq \mathfrak{q}_i$.

Suppose $\mathfrak{q}_i$ is an isolated primary component. Then $A_{\mathfrak{p}_i}$ is an Artin local ring, hence if $\mathfrak{m}_i$ is its maximal ideal we have $\mathfrak{m}_i^r = 0$ for all sufficiently large r, hence $\mathfrak{q}_i = \mathfrak{p}_i^{(r)}$ for all large r.

If $\mathfrak{q}_i$ is an embedded primary component, then $A_{\mathfrak{p}_i}$ is *not* Artinian, hence the powers $\mathfrak{m}_i^r$ are all distinct, and so the $\mathfrak{p}_i^{(r)}$ are all distinct. Hence in the given primary decomposition we can replace $\mathfrak{q}_i$ by any of the infinite set of $\mathfrak{p}_i$-primary decompositions of 0 which differ only in the $\mathfrak{p}_i$-component.

Solution. *Proof.* □

Exercise 2. Let A be a Noetherian ring. Prove that the following are equivalent:

i) A is Artinian;

ii) $\operatorname{Spec}(A)$ is discrete and finite;

iii) $\operatorname{Spec}(A)$ is discrete

Solution. *Proof.* □

Exercise 3. Let k be a field and A a finitely generated k-algebra. Prove that the following are equivalent:

i) A is Artinian;

ii) A is a finite k-algebra.

Solution. *Proof.* □

[To prove that (i)⇒(ii), use (8.7) to reduce to the case where A is an Artin local ring. By the Nullstellensatz, the residue field of A is a finite extension of k. Now use the fact that A is of finite length as an A-module. To prove (ii)⇒(i), observe that the ideals of A are k-vector subspaces and therefore satisfy d.c.c.]

Exercise 4. Let $f\ A \to B$ be a ring homomorphism of finite type. Consider the following statements:

i) f is finite;

ii) the fibres of f^* are discrete subspaces of $\mathrm{Spec}(B)$;

iii) for each prime ideal $\mathfrak{p}$ of A, the ring $B \otimes_A k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$-algebra. ($k(\mathfrak{p})$ is the residue field of $A_{\mathfrak{p}}$);

iv) the fibres of f^* are finite.

Prove that (i)⇒ (ii)⇔ (iii)⇒ (iv). [Use Exercises 2 and 3.]

Is f is integral and the fibres of f^* are finite, is f necessarily finite?

Solution. *Proof.* □

Exercise 5. In Chapter 5, Exercise 16, show that X is a finite covering of L (i.e., the number of points of X lying over a given point of L is finite and bounded).

Solution. *Proof.* □

Exercise 6. Let A be a Noetherian ring and $\mathfrak{q}$ a $\mathfrak{p}$-primary ideal in A. Consider chains of primary ideals from $\mathfrak{q}$ to $\mathfrak{p}$. Show that all such chains are of finite bounded length, and that all maximal chains have the same length.

Solution. *Proof.* □

Chapter 9

Discrete Valuation Rings and Dedekind Domains

Exercises

Exercise 1. Let A be a Dedekind domain, S a multiplicatively closed subset of A. Show that $S^{-1}A$ is either a Dedekind domain or the field of fractions of A.

Suppose that $S \neq A \setminus \{0\}$, and let H, H' be the ideal class groups of A and $S^{-1}A$ respectively. Show that extension of ideals induces a surjective homomorphism $H \to H'$.

Solution. *Proof.* □

Exercise 2. Let A be a Dedekind domain. If $f = a_0 + a_1x + \cdots + a_nx^n$ is a polynomial with coefficients in A, the *content* of f is the ideal $c(f) = (a_0, \ldots, a_n)$ in A. Prove *Gauss's lemma* that $c(fg) = c(f)c(g)$.

[Localize at each maximal ideal.]

Solution. *Proof.* □

Exercise 3. A valuation ring (other than a field) is Noetherian if and only if it is a discrete valuation ring.

Solution. *Proof.* □

Exercise 4. Let A be a local domain which is not a field and in which the maximal ideal $\mathfrak{m}$ is principal and $\cap_{n=1}^{\infty} \mathfrak{m}^n = 0$. Prove that A is a discrete valuation ring.

Solution. *Proof.* □

Exercise 5. Let M be a finitely-generated module over a Dedekind domain. Prove that M is flat $\Leftrightarrow$ M is torsion-free.

[Use Chapter 3, Exercise 13 and Chapter 7, Exercise 16.]

Solution. *Proof.* □

Exercise 6. Let M be a finitely-generated torsion module ($T(M) = M$) over a Dedekind domain A. Prove that M is uniquely representable as a finite direct sum of modules $A/\mathfrak{p}_i^{n_i}$, where $\mathfrak{p}_i$ are nonzero prime ideals of A. [For each $\mathfrak{p} \neq 0$, $M_{\mathfrak{p}}$ is a torsion $A_{\mathfrak{p}}$-module; use the structure theorem for modules over a principal ideal domain.]

Solution. *Proof.* □

Exercise 7. Let A be a Dedekind domain and $\mathfrak{a} \neq 0$ an ideal in A. Show that every ideal in $A/\mathfrak{a}$ is principal.

Deduce that every ideal in A can be generated by at most 2 elements.

Solution. *Proof.* □

Exercise 8. Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ be three ideals in a Dedekind domain. Prove that

$$\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} \cap \mathfrak{b}) + (\mathfrak{a} \cap \mathfrak{c})$$
$$\mathfrak{a} + (\mathfrak{b} \cap \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{c}).$$

[Localize.]

Solution. *Proof.* □

Exercise 9. (Chinese Remainder Theorem). Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals and let $x_1, \ldots, x_n$ be elements in a Dedekind domain A. Then the system of congruences $x \equiv x_i \pmod{\mathfrak{a}_i}$ $(1 \leq i \leq n)$ has a solution x in A $\Leftrightarrow$ $x_i \equiv x_j \pmod{\mathfrak{a}_i + \mathfrak{a}_j}$ whenever $i \neq j$.

[This is equivalent to saying that the sequence of A-modules

$$A \xrightarrow{\phi} \oplus_{i=1}^{n} A/\mathfrak{a}_i \xrightarrow{\psi} \oplus_{i<j} A/(a_i + \mathfrak{a}_j)$$

is exact, where ϕ and ψ are defined as follows:
$\phi(x) = (x + \mathfrak{a}_1, \ldots, x + \mathfrak{a}_n)$; $\psi(x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n)$ has (i, j)-component $x_i - x_j + \mathfrak{a} + \mathfrak{a}_j$. To show that this sequence is exact it is enough to show that it is exact when localized at any $\mathfrak{p} \neq 0$: in other words we may assume that A is a discrete valuation ring, and then it is easy.]

Solution. *Proof.* □

Chapter 10

Completions

Exercises

Exercise 1. Let $\alpha_n : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ be the injection of abelian groups given by $\alpha_n(1) = p^{n-1}$,and let $\alpha : A \to B$ be the direct sum of all the α_n (where A is a countable direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$, and B is the direct sum of the $\mathbb{Z}/p^n\mathbb{Z}$). Show that the p-adic completion of A is just A but that the completion of A for the topology induced from the p-adic topology on B is the direct *product* of the $\mathbb{Z}/p\mathbb{Z}$. Deduce that p-adic completion is *not* a right-exact functor on the category of all $\mathbb{Z}$-modules.

Solution. *Proof.* □

Exercise 2. In Exercise 1, let $A_n = \alpha^{-1}(p^n B)$, and consider the exact sequence

$$0 \to A_n \to A \to A/A_n \to 0.$$

Show that $\varprojlim$ is not right exact, and compute $\varprojlim^1 A_n$.

Solution. *Proof.* □

Exercise 3. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal and M a finitely-generated A-module. Using Krull's Theorem and Exercise 14 of Chapter 3, prove that

$$\bigcap_{n=1}^{\infty} \mathfrak{a}^n M = \bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \operatorname{Ker}(M \to M_m),$$

where $\mathfrak{m}$ runs over all maximal ideals containing $\mathfrak{a}$.

Deduce that

$$\hat{M} = 0 \Leftrightarrow \mathrm{Supp(M)} \cap V(\mathfrak{a}) = \emptyset \qquad (\text{in } \mathrm{Spec}(A)).$$

[The reader should think of $\hat{M}$ as the "Taylor expansion" of M transversal to the subscheme $V(\mathfrak{a})$: the above result then show that M is determined in a neighborhood of $V(\mathfrak{a})$ by its Taylor expansion.]

Solution. *Proof.* □

Exercise 4. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal in A, and $\hat{A}$ the $\mathfrak{a}$-adic completion. For any $x \in A$, let $\hat{x}$ be the image of x in $\hat{A}$. Show that

$$x \text{ is not a zero-divisor in } A \Rightarrow \hat{x} \text{ is not a zero-divisor in } \hat{A}.$$

Does this imply that

$$A \text{ is an integral domain} \Rightarrow \hat{A} \text{ is an integral domain?}$$

Solution. *Proof.* □

Exercise 5. Let A be a Noetherian ring and let $\mathfrak{a}$, $\mathfrak{b}$ be ideals in A. If M is any A-module, let $M^{\mathfrak{a}}$, $M^{\mathfrak{b}}$ denote its $\mathfrak{a}$-adic and $\mathfrak{b}$-adic, respectively. If M is finitely generated, prove that $(M^{\mathfrak{a}})^{\mathfrak{b}} \cong (M^{\mathfrak{a}+\mathfrak{b}}$.

[Take the $\mathfrak{a}$-completion of the exact sequence

$$0 \to \mathfrak{b}^m M \to M \to M/\mathfrak{b}^m M \to 0$$

and apply **??**. then use the isomorphism

$$\varprojlim_{\mathfrak{m}} \left(\varprojlim_{\mathfrak{n}} M/(\mathfrak{a}^n M + \mathfrak{b}^m M) \right) \cong \varprojlim M/(\mathfrak{a}^n M + \mathfrak{b}^n M)$$

and the inclusion $(\mathfrak{a}+\mathfrak{b})^{2n} \subseteq \mathfrak{a}^n + \mathfrak{b}^n \subset (\mathfrak{a}+\mathfrak{b})^n$.]

Solution. *Proof.* □

Exercise 6. Let A be a Noetherian ring and $\mathfrak{a}$ an ideal in A. Prove that $\mathfrak{a}$ is contained in the Jacobson radical of A if and only if every maximal ideal of A is closed for the $\mathfrak{a}$-topology. (A Noetherian topological ring in which the topology is defined by an ideal contained in the Jacobson radical is called a *Zariski ring.* Examples are local rings and (by **??**(iv)) $\mathfrak{a}$-adic completions.)

Solution. *Proof.* □

Exercise 7. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, and $\hat{A}$ the $\mathfrak{a}$-adic completion. Prove that $\hat{A}$ is faithfully flat over A (Chapter 3, Exercise 16) if and only if A is a Zariski ring (for the $\mathfrak{a}$-topology).
[Since $\hat{A}$ is flat over A, it is enough to show that

$$M \to \hat{M} \text{ injective for all finitely generated } M \Leftrightarrow A \text{ is Zariski;}$$

now use (**??**) and Exercise 6.

Solution. *Proof.* □

Exercise 8. Let A be the local ring of the origin in $\mathbb{C}^n$ (i.e., the ring of all rational functions $f/g \in \mathbb{C}(z_1, \ldots, z_n)$ with $g(0) \neq 0$), let B be the ring of power series in $z_1, \ldots, z_n$ which converge in some neighborhood of the origin, and let C be the ring of formal power series in $z_1, \ldots, z_n$, so that $A \subset B \subset C$. Show that B is a local ring and that its completion for the maximal ideal topology is C. Assuming that B is Noetherian, prove that B is A-flat. [Use Chapter 3, Exercise 17, and Exercise 7 above.]

Solution. *Proof.* □

Exercise 9. Let A be a local ring, $\mathfrak{m}$ its maximal ideal. Assume that A is $\mathfrak{m}$-adically complete. For any polynomial $f(x) \in A[x]$, let $\overline{f}(x) \in (A/\mathfrak{m})[x]$ denote its reduction mod $\mathfrak{m}$. Prove *Hensel's lemma:* if $f(x)$ is monic of degree n and if there exist coprime monic polynomials $\overline{g}(x)$, $\overline{h}(x) \in (A/\mathfrak{m})[x]$ of degrees r, $n-r$ with $\overline{f}(x) = \overline{g}(x)\overline{h}(x)$, then we can lift $\overline{g}(x)$, $\overline{h}(x)$ back to monic polynomials $g(x)$, $h(x) \in A[x]$ such that $f(x) = g(x)h(x)$. [Assume inductively that we have constructed $g_k(x)$, $h_k(x) \in A[x]$ such that $g_k(x)h_k(x) - f(x) \in \mathfrak{m}^k A[x]$. Then use the fact that since $\overline{g}(x)$ and $\overline{h}(x)$ are coprime we can find $\overline{a}_p(x)$ and $\overline{b}_p(x)$, of degrees $\leq n-r$, r respectively, such that $x^p = \overline{a}_p(x)\overline{g}_k(x) + \overline{b}_p(x)\overline{h}_k(x)$, where p is any integer such that $1 \leq p \leq n$ Finally, use the completeness of A to show that the sequences $g_k(x)$, $h_k(x)$ converge to the required $g(x)$, $h(x)$.]

Solution. *Proof.* □

Exercise 10.

i) With the notation of Exercise 9, deduce from Hensel's lemma that if $\overline{f}(x)$ has a simple root $\alpha \in A/\mathfrak{m}$, then $f(x)$ has a simple root $a \in A$ such that $\alpha = a \mod \mathfrak{m}$.

ii) Show that 2 is a square in the ring of 7-adic integers.

iii) Let $f(x, y) \in k[x, y]$, where k is a field, and assume that $f(0, y)$ has $y = a_0$ as a simple root. Prove that there exists a formal power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $f(x, y(x)) = 0$. (This gives the "analytic branch" of the curve $f = 0$ through the point $(0, a_0)$.)

Solution. *Proof.* □

Exercise 11. Show that the converse of (10.26) is false, even if we assume that A is local and that $\hat{A}$ is a finitely-generated A-module. [Take A to be the ring of germs of C^∞ functions of x at $x = 0$, and use Borel's Theorem that every power series occurs as the Taylor expansion of some C^∞ function.]

Solution. *Proof.* □

Exercise 12. If A is Noetherian, then $A[[x_1, \ldots, x_n]]$ is a faithfully flat A-algebra. [Express $A \to A[[x_1, \ldots, x_n]]$ as a composition of flat extensions, and use Exercise 5(v) of Chapter 1.]

Solution. *Proof.* □

Chapter 11

Dimension Theory

Exercises

Exercise 1. Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial over an algebraically closed field k. A point P on the variety $f(x) = 0$ is *non-singular* $\Leftrightarrow$ not all partial derivatives $\partial f / \partial x_i$ vanish at P. Let $A = k[x_1, \ldots, x_n]/(f)$, and let $\mathfrak{m}$ be the maximal ideal of A corresponding to the point P. Prove that P is non-singular $\Leftrightarrow$ $A_\mathfrak{m}$ is a regular local ring.
[By (11.18) we have $\dim (A_\mathfrak{m}) = n - 1$. Now

$$\mathfrak{m}/\mathfrak{m}^2 \cong (x_1, \ldots, x_n)/(x_1, \ldots, x_n)^2 + (f)$$

and has dimension $n-1$ if and only if $f \notin (x_1, \ldots, x_n)^2$.]

Solution. *Proof.* □

Exercise 2. In (11.21) assume that A is complete. Prove that the homomorphism
$k[[t_1, \ldots, t_d]] \to A$ given by $t_i \mapsto x_i$ $(1 \le i \le d)$ is injective and that A is a finitely-generated module over $k[[t_1, \ldots, t_d]]$. [Use (10.24).]

Solution. *Proof.* □

Exercise 3. Extend (11.25) to non-algebraically-closed field. [If $\overline{k}$ is the algebraic closure of k, then $\overline{k}[x_1, \ldots, x_n]$ is integral over $k[x_1, \ldots, x_n]$.]

Solution. *Proof.* □

Exercise 4. An example of a Noetherian domain of infinite dimension (Nagata). Let k be a field and let $A = k[x_1, x_2, \ldots, x_n, \ldots]$ be a polynomial ring over k in a countably infinite set of indeterminates. Let $m_1, m_2, \ldots$ be and increasing sequence of positive integers such that $m_{i+1} - m_i > m_i - m_{i-1}$ for all $i > 1$. Let $\mathfrak{p}_i = (x_{m_i+1}, \ldots x_{m_{i+1}})$ and let S be the complement in A of the union of the ideals $\mathfrak{p}_i$.

Each $\mathfrak{p}_i$ is a prime ideal and therefore the set S is multiplicatively closed. The ring $S^{-1}A$ is Noetherian by Chapter 7, Exercise 9. Each $S^{-1}\mathfrak{p}_i$ has height equal to $m_{i+1} - m_i$, hence $\dim\left(S^{-1}A\right) = \infty$.

Solution. *Proof.* □

Exercise 5. Reformulate (11.1) in terms of the Grothendieck group $K(A_0)$ (Chapter 7, Exercise 25).

Solution. *Proof.* □

Exercise 6. Let A be a ring (not necessarily Noetherian). Prove that

$$1 + \dim(A) \leq \dim(A[x]) \leq 1 + 2\dim(A).$$

[Let $f : A \to A[x]$ be the embedding and consider the fiber $f^* : \mathrm{Spec}(A[x]) \to \mathrm{Spec}(A)$ over a prime ideal $\mathfrak{p}$ of A. This fiber can be identified with the spectrum of $k \otimes_A A[x] \cong k[x]$, where k is the residue field of $\mathfrak{p}$ (Chapter 3, Exercise 21), and $\dim(k[x]) = 1$. Now use Exercise 7(ii) of Chapter 4.]

Solution. *Proof.* □

Exercise 7. Let A be a Noetherian ring. Then

$$\dim(A[x]) = 1 + \dim(A),$$

and hence, by induction on n,

$$\dim(A[x_1, \ldots, x_n]) = n + \dim(A).$$

[Let $\mathfrak{p}$ be a prime ideal of height m in A. Then there exist $a_1, \ldots, a_m \in \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime ideal belonging to the ideal $\mathfrak{a} = (a_1, \ldots, a_m)$. By Exercise 7 of Chapter 4, $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$ and therefore height $\mathfrak{p}[x] \leq m$. On the other hand, a chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m = \mathfrak{p}$ gives rise to a chain $\mathfrak{p}_0[x] \subset \cdots \subset \mathfrak{p}_m[x] = \mathfrak{p}[x]$, hence height $\mathfrak{p}[x] \geq m$. Hence height $\mathfrak{p}[x] =$ height $\mathfrak{p}$. Now use the argument of Exercise 6.]

Solution. *Proof.* □